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PROJECT SQUID

TECHNICAL REPORT ARAP-7-P-2

THE FOUNDATIONS OF NONEQUILIBRIUM STATISTICAL MECHANICS

Vol. 2 of 2 Vols.

By

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DEPARTMENT OF AEROSPACE ENGINEERING
SCHOOL OF ENGINEERING AND APPLIED SCIENCE
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OFFICE OF NAVAL RESEARCH, DEPARTMENT OF THE NAVY

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THE FOUNDATIONS OF NONEQUILIBRIUM STATISTICAL MECHANICS*

Vol. 2 of 2 Vols.

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Guido Sandri

Aeronautical Research Associates of Princeton, Inc.
Princeton, New Jersey

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PROJECT SQUID HEADQUARTERS
DEPARTMENT OF AEROSPACE ENGINEERING
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ABSTRACT

In the second part of this paper we complete the program set forth in the first part (1). We continue the discussion of the kinetic expansions of the Liouville equation. Finally we introduce and discuss the "superkinetic" expansions. The numbering of sections continues that of the previous paper.

SECTION 9

THE SHORT-RANGE THEORY

We consider the hierarchy (6.15) with the choice of parameters (6.17), that is we assume:

$$n r_o^3 = \epsilon << 1, \quad \frac{\phi_o}{m v_{th}^2} \simeq 1 \tag{9.1}$$

Therefore,

$$\frac{\partial F^s}{\partial t} + \mathcal{H}^s F^s = \epsilon L_s F^{s+1} \tag{9.2}$$

Since the perturbation expansions:

$$F^1 = f^0 + \varepsilon f^1 + \varepsilon^2 f^2 + \ldots \qquad s = 1 \tag{9.3}$$

$$F^s = F^{s0} + \varepsilon F^{s1} + \varepsilon^2 F^{s2} + \ldots \qquad s \neq 1 \tag{9.4}$$

give, just a for a weakly-coupled gas, a linear growth with time we choose an extension:

$$f^k \Rightarrow \underline{f}^k, \quad \underline{F}^{sk} \Rightarrow \underline{F}^{sk} \tag{9.5}$$

with coordinates for the embedded domain:

$$\tau_0 = t, \ \tau_1 = \epsilon t, \ \ldots \tau_n = \epsilon^n t, \ldots \tag{9.6}$$

From (9.2) we find accordingly, for s = 1

$$\frac{\partial \underline{f}^0}{\partial \tau_0} = 0 \tag{9.7}$$

$$\frac{\partial \underline{f}^1}{\partial \tau_0} + \frac{\partial \underline{f}^0}{\partial \tau_1} = L\,\underline{F}^{20} \tag{9.8}$$

$$\frac{\partial \underline{f}^2}{\partial \tau_0} + \frac{\partial \underline{f}^1}{\partial \tau_1} + \frac{\partial \underline{f}^0}{\partial \tau_2} = L\,\underline{F}^{21} \tag{9.9}$$

$$\frac{\partial \underline{f}^3}{\partial \tau_0} + \frac{\partial \underline{f}^2}{\partial \tau_1} + \frac{\partial \underline{f}^1}{\partial \tau_2} + \frac{\partial \underline{f}^0}{\partial \tau_3} = L\,\underline{F}^{22} \tag{9.10}$$

For the two-body distribution function we have,

$$\frac{\partial \underline{F}^{20}}{\partial \tau_0} + \mathcal{H}^2 \underline{F}^{20} = 0 \tag{9.11}$$

$$\frac{\partial \underline{F}^{21}}{\partial \tau_0} + \mathcal{H}^2 \underline{F}^{21} = -\frac{\partial \underline{F}^{20}}{\partial \tau_1} + L_2\,\underline{F}^{30} \tag{9.12}$$

$$\frac{\partial \underline{F}^{22}}{\partial \tau_0} + \mathcal{H}^2 F^{22} = -\frac{\partial \underline{F}^{21}}{\partial \tau_1} - \frac{\partial F^{20}}{\partial \tau_2} + L_2\,F^{31} \tag{9.13}$$

For the three-body distribution function:

$$\frac{\partial F^{30}}{\partial \tau_0} + \mathcal{H}^3 \underline{F}^{30} = 0 \tag{9.14}$$

$$\frac{\partial \underline{F}^{31}}{\partial \tau_0} + \mathcal{H}^3 F^{31} = -\frac{\partial \underline{F}^{30}}{\partial \tau_1} + L_3\,\underline{F}^{40} \tag{9.15}$$

It will be clear later that we are carrying the approximation to one order higher than it is physically meaningful to do so. The purpose of such a calculation is to show the breakdown of the asymptotic series. It will be useful to keep in mind the orbit equations:

$$\underline{x}_i(t) = e^{\mathcal{H}^s t}\,\underline{x}_i(0) \tag{9.16}$$

$$\underline{v}_i(t) = e^{\mathcal{H}^s t}\,\underline{v}_i(0) \tag{9.17}$$

which correspond to Hamilton's equations

$$\dot{\underline{x}}_i = \mathcal{H}^s\,\underline{x}_i \tag{9.18}$$

$$\dot{\underline{v}}_i = \mathcal{H}^s\,\underline{v}_i \tag{9.19}$$

The orbit equation can be written in another form if we use

$$\underline{x}_i(t) = \underline{x}_i(0) + \oint_0^t \underline{v}_i(\lambda)\,d\lambda \tag{9.20}$$

$$\underline{v}_i(t) = \underline{v}_i(0) + \oint_0^t \dot{\underline{v}}_i(\lambda)\,d\lambda \tag{9.21}$$

where $\oint$ denotes integration along the orbit. Since we have, by (9.21):

$$\begin{aligned}\dot{\underline{v}}_i(t) &= \mathcal{H}^s e^{\mathcal{H}^s t}\,\underline{v}_i(0)\\ &= e^{\mathcal{H}^s t}\,\dot{\underline{v}}_i(0)\end{aligned} \tag{9.22}$$

substitution of (9.17) and (9.22) into (9.20) and (9.21) respectively gives:

$$\underline{x}_i(t) = \underline{x}_i(0) + \frac{e^{\mathcal{H}^s t} - 1}{\mathcal{H}^s}\,\underline{v}_i(0) \tag{9.23}$$

$$\underline{v}_i(t) = \underline{v}_i(0) + \frac{e^{\mathcal{H}^s t} - 1}{\mathcal{H}^s}\,\dot{\underline{v}}_i(0) \tag{9.24}$$

Equation (9.23) constitutes the formal solution to the s-body problem.

A. The Simple Initial Value Problem

The definition of this problem is identical to the one adapted in the weak-coupling expansion.

(i) Zeroth-Order Theory

The one-body function is constant:

$$\underline{f}^{o}(\tau_0) = \underline{f}^{o} \tag{9.25}$$

For the s-body function we have, from (9.11) and (9.14):

$$\underline{F}^{so}(\tau_0) = e^{-\mathcal{H}^s \tau_0} \prod^{s} \underline{f}^{o} \tag{9.26}$$

which expresses the fact that initially uncorrelated particles are very quickly correlated by collisions. We adopt the notation:

$$S^s(\tau) = e^{+\mathcal{H}^s \tau} \tag{9.27}$$

with the convention $S^s \equiv S^s(-\infty)$. We can therefore rewrite (9.26) as

$$\underline{F}^{so}(\tau_0) = S^s(-\tau_0) \prod^{s} \underline{f}^{o} \underset{\tau_0}{\approx} S^s \prod^{s} \underline{f}^{o} \tag{9.28}$$

(ii) First-Order Theory (Boltzmann's Equation)

Substituting (9.26) into (9.8) we find

$$\frac{\partial \underline{f}^1}{\partial \tau_0} + \frac{\partial \underline{f}^0}{\partial \tau_1} = L e^{-\mathcal{H}^2 \tau_0} \underline{f}^0 \underline{f}^0 \underset{\tau_0}{\approx} L S^2 \underline{f}^0 \underline{f}^0 \qquad (9.29)$$

Therefore, in view of (9.25)

$$\frac{\partial \underline{f}^0}{\partial \tau_1} = L S^2 \underline{f}^0 \underline{f}^0 \qquad (9.30)$$

$\underline{f}^1$ remains as a transient; schematically (where (10.6) is already implied):

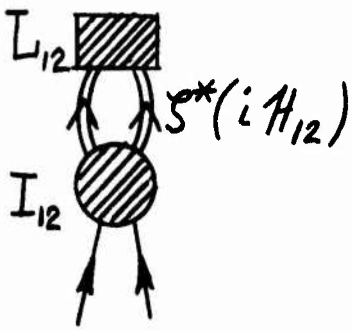


Fig. 14.

This is Bogolubov's form of Boltzmann's equation. It can be transformed into the familiar Boltzmann form (6.25) if one keeps in mind that for purely repulsive two-body potentials there is no binding and therefore

$$\partial S^2(t)/\partial t \underset{t \to \infty}{\longrightarrow} 0 \qquad (9.31)$$

This asymptotic condition gives the useful identity

$$K^s S^s = I^s S^s \tag{9.32}$$

One then must perform Bogolubov's "cylindrical" integration. The geometry of the collision is shown in Figure 15.

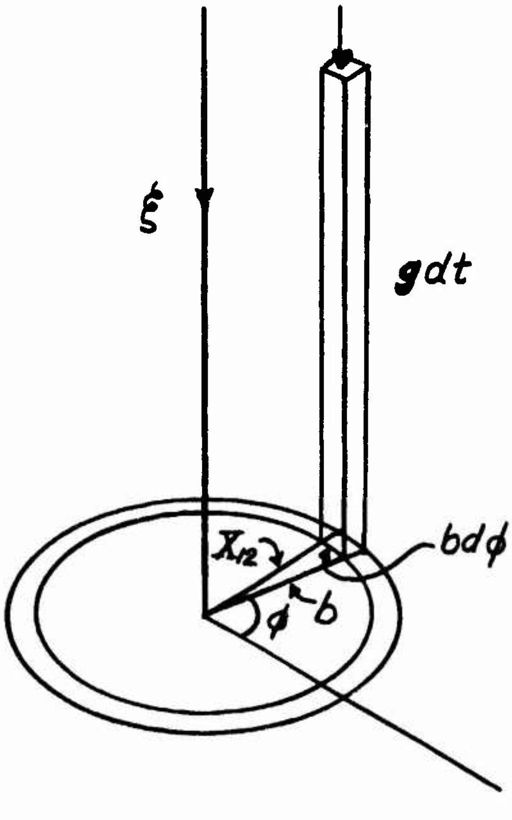


Fig. 15.

ξ is the axis of the integration, $\underline{g} \equiv \underline{V}_1 - \underline{V}_2$ ..
Figure 15 shows the connection between the kinetic argument and the statistical dynamical one.

From (9.12) we have, using (9.30)

$$\frac{\partial \underline{E}^{21}}{\partial \tau_0} + \mathcal{H}^2 \underline{E}^{21} = -e^{-\mathcal{H}^2 \tau_0}\left[L_{13} S_{13} + L_{23} S_{23}\right] \overset{3}{\pi} \underline{f}^{\circ} + L_2 e^{-\mathcal{H}^3 \tau_0} \overset{3}{\pi} \underline{f}^{\circ}$$

$$\underset{\tau_0}{\sim} \left\{-S_{12}\left[L_{13} S_{13} + L_{23} S_{23}\right] + L_2 S_{123}\right\} \overset{3}{\pi} \underline{f}^{\circ} \tag{9.33}$$

which gives

$$\underline{E}^{21} \xrightarrow[\tau_0 \to \infty]{} \int_0^{\infty} e^{-\mathcal{H}^2 \lambda} d\lambda \left\{-S_{12}\left[L_{13} S_{13} + L_{23} S_{23}\right] + L_2 S_{123}\right\} \overset{3}{\pi} \underline{f}^{\circ} \tag{9.34}$$

This formula is improper because it contains a <u>divergence</u>. Its importance will be clear shortly.

From (9.15), we find for the three-body distribution:

$$\frac{\partial \underline{E}^{31}}{\partial \tau_0} + \mathcal{H}^3 \underline{E}^{31} = -e^{-\mathcal{H}^3 \tau_0}\left[L_{14} S_{14} + L_{24} S_{24} + L_{34} S_{34}\right] \overset{4}{\pi} \underline{f}^{\circ} + L_3 e^{-\mathcal{H}^4 \tau_0} \overset{4}{\pi} \underline{f}^{\circ}$$

$$\underset{\tau_0}{\sim} \left\{-S^3\left[L_{14} S_{14} + L_{24} S_{24} + L_{34} S_{34}\right] + L_3 S^4\right\} \overset{4}{\pi} \underline{f}^{\circ} \tag{9.35}$$

(iii) <u>Second-Order Theory</u>

From (9.9) we have, with (9.34)

$$\frac{\partial \underline{f}^2}{\partial \tau_0} + \frac{\partial \underline{f}^1}{\partial \tau_1} + \frac{\partial \underline{f}^{\circ}}{\partial \tau_2} \underset{\tau_0}{\sim}$$

$$\underset{\tau_0}{\sim} L\, \zeta^*(i\mathcal{H}^2)\left\{-S_{12}\left[L_{13} S_{13} + L_{23} S_{23}\right] + L_2 S^3\right\} \overset{4}{\pi} \underline{f}^{\circ} \tag{9.36}$$

where we have introduced the operator valued distribution

$$\zeta^*(i\mathcal{H}^s) = \int_0^{\infty} e^{-\mathcal{H}^s\lambda}\, d\lambda \tag{9.37}$$

We shall also consider:

$$\delta(i\mathcal{H}^s) = \int_{-\infty}^{+\infty} e^{-\mathcal{H}^s\lambda}\, d\lambda \tag{9.38}$$

From (9.35) if f^2 is to remain bounded

$$\frac{\partial f^1}{\partial \tau_1} + \frac{\partial f^0}{\partial \tau_2} = L\,\zeta^*(i\mathcal{H}^2)\left\{-S_{12}\left[L_{13}S_{13} + L_{23}S_{23}\right] + L_2 S^3\right\}\frac{3}{\Pi} f^0 \tag{9.39}$$

The right hand side is schematized in Fig. 16.

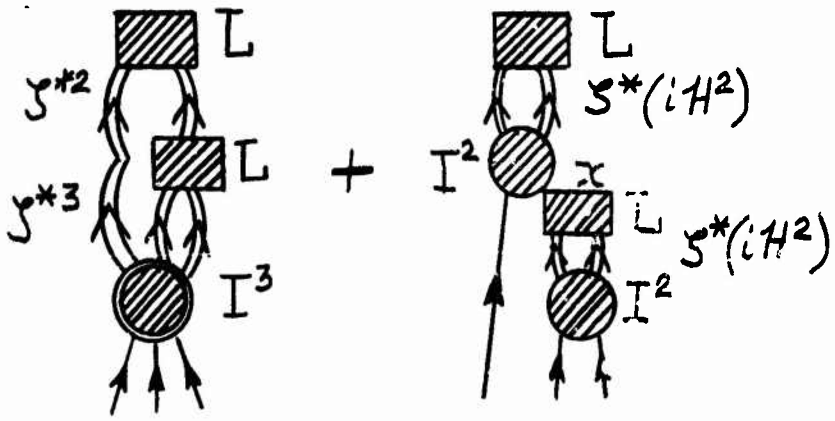


Fig. 16.

(iv) The "Bogolubov Problem" and the Choh-Uhlenbeck Formula

The right hand side of (9.39) is identical to the A^2 of Choh and Uhlenbeck. The Choh-Uhlenbeck formula results, according to them, by the two following steps. Call the curly bracket in (9.39), then

$$
\begin{aligned}
L\zeta^*\Lambda &= \int d\underline{x}_2 d\underline{v}_2\, I^2 \int_0^\infty e^{-\mathcal{H}^2\lambda} d\lambda\, \Lambda \\
&= \int d\underline{x}_2 d\underline{v}_2 \int_0^\infty \left(\mathcal{K}^2 + \frac{\partial}{\partial\lambda}\right) e^{-\mathcal{H}^2\lambda} d\lambda\, \Lambda
\end{aligned}
\tag{9.40}
$$

where the equation of motion of $\exp(-\mathcal{H}^2\lambda)$ has been used. By spatial homogeneity the $\mathcal{K}^2$ contribution is first set equal to zero. According to Choh-Uhlenbeck then the upper limit of the integration does not contribute. Therefore,

$$L\zeta^*\Lambda = -\int d\underline{x}_2 d\underline{v}_2 \Lambda \tag{9.41}$$

It is clear that since f' is a transient, i.e. $\frac{\partial f'}{\partial \tau_0} \underset{\tau_0}{\approx} 0$ one can choose $f'(\tau_0 = 0)$ so as to insure

$$f' \xrightarrow[\tau_0 = \infty]{} 0 \tag{9.42}$$

This choice is different from that of the simple initial value problem. The choice (9.42) generalized to

$$f^K \xrightarrow[\tau_0 \to \infty]{} 0 \qquad K \neq 0 \tag{9.43}$$

and supplemented by

$$e^{-\mathcal{H}^S \tau_0} g^{sK} \xrightarrow[\tau_0 \to \infty]{} 0 \tag{9.44}$$

will be called the "Bogolubov problem". This is closely analogous to the "Enskog problem" of the transition to fluid dynamic (see Section 15).

Referring back to the basic equations of the short-range theory (9.7) to (9.15) we readily see that Bogolubov's problem leads to the following equations. In zeroth order:

$$\underline{f}^{0}(\tau_0) = \underline{f}^{0} \tag{9.45}$$

$$\underline{F}^{s\,0}(\tau_0) \xrightarrow[\tau_0 \to \infty]{} S^{s} \overset{s}{\Pi} \underline{f}^{0} \tag{9.46}$$

In first order:

$$\frac{\partial \underline{f}^{0}}{\partial \tau_1} = L S \underline{f}^{0} \underline{f}^{0} \tag{9.47}$$

which agrees with (9.30) and

$$\underline{f}' \underset{\tau_0}{\sim} 0$$

The Boltzmann equation therefore follows. In addition we have, for the two-body function (9.34). Furthermore, from (9.36) we now conclude

$$\frac{\partial \underline{f}^{0}}{\partial \tau_2} = L S^{*}(iH^{2}) \left\{ -S_{12}\left[L_{13}S_{13} + L_{23}S_{23}\right] + L_2 S^{3} \right\} \overset{3}{\Pi} \underline{f}^{0} \tag{9.48}$$

The Choh-Uhlenbeck theory consists in the restriction:

$$\frac{\partial \underline{f}^{0}}{\partial \tau_0} + \epsilon \frac{\partial \underline{f}^{0}}{\partial \tau_1} + \epsilon^2 \frac{\partial \underline{f}^{0}}{\partial \tau_2} = \epsilon L S \underline{f}^{0} \underline{f}^{0} + \epsilon^2 L S^{*} \left\{ -S_{12}\left[L_{13}S_{13} + L_{23}S_{23}\right] + L_2 S^{3} \right\} \overset{3}{\Pi} \underline{f}^{0}$$

$$\Longrightarrow \frac{\partial f^{0}}{\partial t} = \epsilon L S f^{0} f^{0} + \epsilon^2 L S^{*} \left\{ -S_{12}\left[L_{13}S_{13} + L_{23}S_{23}\right] + L_2 S^{3} \right\} \overset{3}{\Pi} f^{0} \tag{9.49}$$

(v) Conclusion of the Second-Order Theory

We included the discussion of Section (iv) in the second-order theory in view of the special interest it holds and the crucial role it plays in a deeper insight into the kinetic theory.

We now complete the second-order theory. The H-theorem for the Boltzmann equation (9.30) insures that

$$\underline{f}^{\circ}(\tau_1) \xrightarrow[\tau_1 \to \infty]{} \underline{M} \tag{9.50}$$

where $\underline{M}$ is the Maxwellian distribution of thermodynamic equilibrium with density $\underline{\rho}$, mean flow velocity $\underline{u}$ and temperature $\underline{T}$:

$$\underline{M} = \frac{\rho}{(2\pi k \underline{T})^{3/2}} e^{-\frac{(\underline{v}-\underline{u})^2}{2k\underline{T}}} \tag{9.51}$$

If we now analyze (9.39) for large values of τ_1, we have:

$$\frac{\partial \underline{f}'}{\partial \tau_1} + \frac{\partial \underline{M}}{\partial \tau_2} = \angle \zeta^*(iH^2)\left\{-S_{12}\left[\angle_{13} S_{13} + \angle_{23} S_{23}\right] + \angle_2 S^3\right\} \overset{3}{\pi} \underline{M} = 0 \tag{9.52}$$

We can therefore choose

$$\frac{\partial \underline{M}}{\partial \tau_2} = 0 \tag{9.53}$$

and

$$\frac{\partial \underline{f}'}{\partial \tau_1} = \angle \zeta^*(iH^2)\left\{-S_{12}\left[\angle_{13} S_{13} + \angle_{23} S_{23}\right] + \angle_2 S^3\right\} \overset{3}{\pi} \underline{f}^{\circ} \tag{9.54}$$

which can be rewritten as an explicit quadrature:

$$\underline{f}^{1}(\tau_1) = L S^{*}(i\pi/2)\left\{ -S_{12}\left[L_{13}S_{13} + L_{23}S_{23} \right] + L_{2}S^{3} \right\} \int_0^{\tau_1} d\lambda \, \overset{3}{\Pi} \, \underline{f}^{0}(\lambda) \qquad (9.55)$$

where the λ dependence of $\underline{f}^{0}$ must be found from the Boltzmann equation (9.30). Since $\underline{f}^{0}$ is τ_0 independent, we can set:

$$\frac{\partial \underline{f}^{0}}{\partial \tau_2} = 0 \qquad (9.56)$$

This is the first example of closure. This concept will be further discussed in Section 13.

The physical meaning of the choice (9.53) and (9.54) is the following. For a spatially homogeneous gas there are basically only two stages in the evolution towards equilibrium.

In the first or pre-kinetic stage $\underline{F}^{s}(s>1)$ becomes synchronized to $\underline{F}^{1}$ while $\underline{F}^{1}$ itself does not change. This "freezing" of the correlations, which is due to the energy momentum balance in the collisions, occurs on the time scale τ_0 whose unit is r_0/v_{th} i.e. the duration of one collision. The mathematical expression of this synchronization is given by the formulae (9.28) and (9.25).

In the second stage or kinetic stage, the one-body function suffices to characterize the evolution of the system in time. In fact the one-body function satisfies an equation (the Boltzmann equation (9.30)) which determines it independently of any knowledge of the two, three-body distributions. The time scale for the kinetic stage is τ_1 whose unit is

$$\frac{r_0}{v_{th}} \frac{1}{\epsilon} = \frac{r_0}{v_{th}} \frac{1}{n r_0^3} = \frac{\lambda}{v_{th}} \qquad (9.57)$$

where λ is the mean free path. It is clear therefore that the unit of the τ_1 scale is the time between two successive collisions. Boltzmann has shown that a gas that satisfies his kinetic equation tends of necessity to equilibrium:

$$\frac{dH}{d\tau_1} = \frac{d}{d\tau_1}\int \underline{f}^0 \operatorname{Log} \underline{f}^0 \, d\underline{v} \leq 0 \qquad (9.58)$$

with the equal sign implying $\underline{f}^0 = \underline{M}$. On this scale therefore the already synchronized $\underline{F}^s$ functions will reach their equilibrium values.

It is therefore to be expected that after the asymptotic limit has been reached no further evolution can occur in the gas. (This of course does not prevent fluctuations). The "closure" condition (9.53) expresses this basic fact. The method of extension (introduced in Section 3) is responsible for our ability to "pinch off" the asymptotic series at the place where it would be damaging. The fact that $\underline{f}^1$, as given by (9.55), is improper simply expresses the breakdown of the asymptotic expansion.

We now turn to the two-body function in second order. From (9.13)

$$\frac{\partial \underline{F}^{22}}{\partial \tau_0} + \mathcal{H}^2 \underline{F}^{22} \underset{\tau_0}{\sim} -\frac{\partial \underline{F}^{21,aso}}{\partial \tau_1} - \frac{\partial \underline{F}^{20,aso}}{\partial \tau_2} + L_2 \underline{F}^{31,aso} \qquad (9.59)$$

where aso means asymptotic in τ_0. But we have

$$\frac{\partial \underline{F}^{20,aso}}{\partial \tau_2} = \frac{\partial}{\partial \tau_2} S^2 \underline{f}^0 \underline{f}^0 = 0 \qquad (9.60)$$

by (9.56). Also from (9.34):

$$\frac{\partial \underline{E}^{31,\,ao\,0}}{\partial \tau_1} = \zeta_{12}^{*}\left\{-S_{12}\left[L_{13}S_{13}+L_{23}S_{23}\right]+L_2 S^3\right\}\cdot$$

$$\cdot\left\{L_{14}S_{14}+L_{24}S_{24}+L_{34}S_{34}\right\}\overset{4}{\pi}\underline{f}^{0} \tag{9.61}$$

Finally, from (9.35)

$$L_2\underline{E}^{31,\,as\,0}$$

$$=L_2\left\{-S^3\left[L_{14}S_{14}+L_{24}S_{24}+L_{34}S_{34}\right]+L_3 S^4\right\}\overset{4}{\pi}\underline{f}^{0} \tag{9.62}$$

whence, substituting (9.60), (9.61), and (9.62) into (9.59):

$$\underline{F}^{22}\underset{\tau_0}{\sim}\zeta_{12}^{*}\Big(L_2\left\{-S^3\left[L_{14}S_{14}+L_{24}S_{24}+L_{34}S_{34}\right]+L_3S^4\right\}+$$

$$+\left\{S_{12}\left[L_{13}S_{13}+L_{23}S_{23}\right]-L_2S^3\right\}\left\{L_{14}S_{14}+L_{24}S_{24}+L_{34}S_{34}\right\}\Big)\overset{4}{\pi}\underline{f}^{0} \tag{9.63}$$

which represents the effect of four-body collisions on the two-body distribution function.

(iv) Third-Order Theory

From (9.10) we have

$$\frac{\partial \underline{f}^{3}}{\partial \tau_0}+\frac{\partial \underline{f}^{2}}{\partial \tau_1}+\frac{\partial \underline{f}^{1}}{\partial \tau_2}+\frac{\partial \underline{f}^{0}}{\partial \tau_3}\underset{\tau_0}{\sim}$$

$$\underset{\tau_0}{\sim} L\zeta_{12}^{*}\Big(S_{12}\left[L_{13}S_{13}+L_{23}S_{23}\right]\left[L_{14}S_{14}+L_{24}S_{24}+L_{34}S_{34}\right]-$$

$$-2L_2S^3\left[L_{14}S_{14}+L_{24}S_{24}+L_{34}S_{34}\right]+L_2S^4\Big)\overset{4}{\pi}\underline{f}^{0} \tag{9.64}$$

To have a well-behaved $\underline{f}^3$ we require

$$\frac{\partial \underline{f}^2}{\partial \tau_1} + \frac{\partial \underline{f}^1}{\partial \tau_2} + \frac{\partial \underline{f}^0}{\partial \tau_3} = \text{r.h.s. of } (9.64) \tag{9.65}$$

We can now require

$$\frac{\partial \underline{M}}{\partial \tau_3} = 0 \tag{9.66}$$

and also

$$\frac{\partial \underline{f}^{1,\ as\ 1}}{\partial \tau_2} = 0 \tag{9.67}$$

which shows the consistency of the closure conditions.

B. The Complete Initial Value Problem

We now allow arbitrary initial correlations to be present in the gas. From (9.7), again

$$\underline{f}^0(\tau_0) = \underline{f}^0 \tag{9.68}$$

from (9.11),

$$\begin{aligned} \underline{F}^{20}(\tau_0) &= e^{-\mathcal{H}^2 \tau_0} \underline{f}^0 \underline{f}^0 + e^{-\mathcal{H}^2 \tau_0} \underline{g}^0(0) \\ &\underset{\tau_0}{\approx} S[\underline{f}^0 \underline{f}^0 + \underline{g}^0(0)] \end{aligned} \tag{9.69}$$

From (9.14),

$$\begin{aligned} \underline{F}^{30}(\tau_0) &= e^{-\mathcal{H}^3 \tau_0} \prod^3 \underline{f}^0 + e^{-\mathcal{H}^3 \tau_0} \sum \underline{f}^0 \underline{g}^0(0) + \\ &+ e^{-\mathcal{H}^3 \tau_0} \underline{h}^0(0) \end{aligned} \tag{9.70}$$

We obtain therefore in first-order

$$\frac{\partial \underline{f}^{1}}{\partial \tau_0} + \frac{\partial \underline{f}^{0}}{\partial \tau_1} = L e^{-\mathcal{H}^2 \tau_0} \left[\underline{f}^0 \underline{f}^0 + \underline{g}^0(0) \right] \tag{9.71}$$

The necessary and sufficient condition for the validity of Boltzmann's equation is accordingly $(\eta > 0)$

$$L e^{-\mathcal{H}^2 \tau_0} \underline{g}^0(0) \underset{\tau_0}{\sim} \frac{1}{\tau_0^{1+\eta}} \tag{9.72}$$

When (9.72) is fulfilled, the behavior of $\underline{f}^{1}$ is transient as for the simple initial value problem, but of course considerably more complex.

For the two-body function we have, from (9.12):

$$\begin{aligned} \frac{\partial \underline{F}^{21}}{\partial \tau_0} + \mathcal{H}^2 \underline{F}^{21} &= e^{-\mathcal{H}^2 \tau_0} \frac{\partial}{\partial \tau_1} \left[\underline{f}^0 \underline{f}^0 + \underline{g}^0(0) \right] + \\ &+ L_2 e^{-\mathcal{H}^3 \tau_0} \left[\overset{3}{\Pi} \underline{f}^0 + \Sigma \underline{f}^0 \underline{g}^0(0) + \underline{h}^0(0) \right] \underset{\tau_0}{\sim} -S_{12} \frac{\partial}{\partial \tau_1} \left[\underline{f}^0 \underline{f}^0 + \underline{g}^0(0) \right] + \\ &+ L_2 S^3 \left[\overset{3}{\Pi} \underline{f}^0 + \Sigma \underline{f}^0 \underline{g}^0(0) + \underline{h}^0(0) \right] \end{aligned} \tag{9.73}$$

whence

$$\underline{F}^{21} \underset{\tau_0}{\sim} \zeta_{12}^{*} \left\{ -S_{12} \frac{\partial}{\partial \tau_1} \left[\underline{f}^0 \underline{f}^0 + \underline{g}^0(0) \right] + L_2 S^3 \left[\overset{3}{\Pi} \underline{f}^0 + \Sigma \underline{f}^0 \underline{g}^0(0) + \underline{h}^0(0) \right] \right\} \tag{9.74}$$

Substitution of (9.74) into:

$$\begin{aligned} \frac{\partial \underline{f}^{2}}{\partial \tau_0} + \frac{\partial \underline{f}^{1}}{\partial \tau_1} + \frac{\partial \underline{f}^{0}}{\partial \tau_2} &\underset{\tau_0}{\sim} L \zeta_{12}^{*} \left\{ -S_{12} \frac{\partial}{\partial \tau_1} \left[\underline{f}^0 \underline{f}^0 + \underline{g}^0(0) \right] + \right. \\ &\left. + L_2 S^3 \left[\Pi \underline{f} + \Sigma \underline{f}^0 \underline{g}^0(0) + \underline{h}^0(0) \right] \right\} \end{aligned} \tag{9.75}$$

A kinetic condition can be extracted from this equation only if

$$L_{12} e^{-\mathcal{H}_{12}\tau_0} L_2 \zeta^{*3} f^0 g^0(0) \underset{\tau_0}{\sim} \frac{1}{\tau_0^{1+\eta}} \qquad (\eta > 0) \tag{9.76}$$

and

$$L_{12} e^{-\mathcal{H}_{12}\tau_0} L_2 \zeta^{*3} g^0(0) \underset{\tau_0}{\sim} \frac{1}{\tau_0^{1+\eta}} \qquad (\eta > 0) \tag{9.77}$$

These conditions are of interest because they are required by closure.

SECTION 10

THE LANDAU EXPANSION

A. The Expansion

In order to establish the relation between the short range theory and the weak coupling expansion, we shall have to consider the dilute, weakly coupled gas. It is clear from Fig. 10 that if we expand the short range results in powers of

$$\left(\frac{\phi_0}{m v_{th}^2}\right)$$

we will not obtain the formulae for a weakly coupled gas but rather those for a dilute weakly coupled gas. Also, if we expand the weak coupling formulae in the dilution parameter $(n r_0^3)$ we shall find a dilute weakly coupled regime. The expansion of the short range theory in powers of

$$\left(\frac{\phi_0}{m v_{th}^2}\right)$$

will be called the Landau expansion.

The basic formula that we shall employ is (2.12) applied to $\mathcal{H} = \mathcal{K} - \varepsilon I$

$$e^{-(\mathcal{K}-\epsilon I)t} = e^{-\mathcal{K}t} + \epsilon e^{-\mathcal{K}t}\int_0^t \left[e^{+\mathcal{K}\lambda} I e^{-\mathcal{K}\lambda}\right] d\lambda + \epsilon^2 e^{-\mathcal{K}t}\int_0^t \left[e^{\mathcal{K}\lambda} I e^{-\mathcal{K}\lambda}\right]\int_0^\lambda \left[e^{\mathcal{K}\lambda'} I e^{-\mathcal{K}\lambda'}\right] d\lambda' d\lambda + O(\varepsilon^3) \tag{10.1}$$

(i) Zeroth order theory. From (9.25) and (9.26) we find

$$\underline{f}^{\circ}(\tau_0) = \underline{f}^{\circ} \tag{10.2}$$

and

$$\underline{F}^{s0}(\tau_0) = e^{-\kappa^s\tau_0}\overset{s}{\Pi}\underline{f}^0 + \epsilon e^{-\kappa^s\tau_0}\int_0^{\tau_0} e^{\kappa^s\lambda} \overset{s}{I} e^{-\kappa^s\lambda} d\lambda\, \overset{s}{\Pi}\underline{f}^0$$

$$= \overset{s}{\Pi}\underline{f}^0 + \epsilon \frac{1-e^{-\kappa^s\tau_0}}{\kappa^s} I^s \overset{s}{\Pi}\underline{f}^0 \qquad (10.3)$$

$$\underset{\tau_0}{\sim} \overset{s}{\Pi}\underline{f}^0 + \epsilon \zeta^*(i\kappa^s) I^s \overset{s}{\Pi}\underline{f}^0$$

(ii) First-Order Theory.

We rewrite the Boltzmann equation (9.30) exploiting the fact that only the two-body correlation rather than the complete two-body function contributes to the kinetic equation. Thus, for a spatially homogeneous gas:

$$L\underline{F}^2(\tau_0) = L(\underline{F}'\underline{F}' + \underline{g}) = L\underline{g} \qquad (10.4)$$

We have from (9.26):

$$\underline{F}^{20}(\tau_0) = e^{-\mathcal{H}^2\tau_0}\underline{f}^0\underline{f}^0$$

$$= \underline{f}^0\underline{f}^0 + \underline{g}^0(\tau_0) \qquad (10.5)$$

Whence

$$\underline{g}^0(\tau_0) = (e^{-\mathcal{H}^2\tau_0} - 1)\underline{f}^0\underline{f}^0 = -\frac{1-e^{-\mathcal{H}^2\tau_0}}{\mathcal{H}^2}\mathcal{H}^2\underline{f}^0\underline{f}^0$$

$$= \frac{1-e^{-\mathcal{H}^2\tau_0}}{\mathcal{H}^2} I^2\underline{f}^0\underline{f}^0 \underset{\tau_0}{\sim} \zeta^*(i\mathcal{H}^2) I^2\underline{f}^0\underline{f}^0 \qquad (10.6)$$

where we have used

$$\mathcal{K}^2 \underline{f}^0 \underline{f}^0 = 0 \tag{10.7}$$

For the Boltzmann equation (9.30) we have therefore

$$\begin{aligned} \frac{\partial \underline{f}^0}{\partial \tau_1} &= L \zeta^*(i \mathcal{H}^2) I^2 \underline{f}^0 \underline{f}^0 \\ &= L \zeta^*(i \mathcal{K}^2) I^2 \underline{f}^0 \underline{f}^0 + O(\epsilon) \end{aligned} \tag{10.8}$$

The result is the small momentum transfer equation of Landau. This result also follows from (10.3).

The Landau expansion for the function $\underline{F}^{21}$ is most readily obtained from the equation (9.12) that it satisfies. Thus, we can write

$$\frac{\partial \underline{F}^{21}}{\partial \tau_0} + \mathcal{K}^2 \underline{F}^{21} - \varepsilon I^2 \underline{F}^{21} =$$

$$= \epsilon \left\{ e^{-\mathcal{H}^2 \tau_0} \left[L_{13} \zeta^*(i \mathcal{H}_{13}) I_{13} + L_{23} \zeta^*(i \mathcal{H}_{23}) I_{23} \right] + L_2 \frac{1 - e^{-\mathcal{H}^3 \tau_0}}{\mathcal{H}^3} I^3 \overset{3}{\Pi} \underline{f}^0 \right. \tag{10.9}$$

We now let, simply

$$\underline{F}^{21} = \underline{F}^0 + \varepsilon \underline{F}^1 + \varepsilon^2 \underline{F}^2 + \ldots \tag{10.10}$$

And, substituting into (10.9):

$$\frac{\partial \underline{F}^0}{\partial \tau_0} + \mathcal{K}^2 \underline{F}^0 = 0 \tag{10.11}$$

This gives

$$F^0 = 0 \tag{10.12}$$

to lowest order.

The first order equation is

$$\frac{\partial \underline{F}^1}{\partial \tau_0} + \kappa^2 \underline{F}^1 = \Big\{ -\big[L_{13}\zeta^*(i\kappa_{13}) I_{13} + L_{23}\zeta^*(i\kappa_{23}) + L_2 \frac{1-e^{-\kappa^3\tau_0}}{\kappa^3}\big(I_{12}+I_{13}+I_{23}\big)\Big\} \overset{3}{\pi}\underline{f}^0 \underset{\tau_0}{\sim} \big[L_{13}(\zeta^* I)_{23} + L_{23}(\zeta^* I)_{13}\big]\overset{3}{\pi}\underline{f}^0 \tag{10.13}$$

which gives:

$$\underline{F}^1 \underset{\tau_0}{\sim} \zeta^*(i\kappa^3)\big[L_{13}\zeta^*(i\kappa_{23}) I_{23} + L_{23}\zeta^*(i\kappa_{13}) I_{13}\big]\overset{3}{\pi}\underline{f}^0 \tag{10.14}$$

which corresponds to the second term of (7.87). The first term is the second Landau approximation for $\underline{F}^{20}$.

To second order we find:

$$\frac{\partial \underline{F}^2}{\partial \tau_0} + \kappa^2 \underline{F}^2 - I^2 \underline{F}^1 = e^{-\kappa^2\tau_0}\int_0^{\tau_0} e^{\kappa^2\lambda} I^2 d\lambda \big[L_{13}(\zeta^* I)_{23} + L_{23}(\zeta^* I)_{13}\big]\overset{3}{\pi}\underline{f}^0 - \\ -\Big\{ L_{13}\int_0^{\infty} e^{-\kappa_{13}\lambda}\int_0^{\lambda} e^{+\kappa_{13}\lambda'} I_{13} e^{-\kappa_{13}\lambda'} I_{13}\, d\lambda\, d\lambda' + \\ + L_{23}\int_0^{\infty} e^{-\kappa_{23}\lambda}\int_0^{\lambda} e^{\kappa_{23}\lambda'} I_{23} e^{-\kappa_{23}\lambda} I_{23}\, d\lambda\, d\lambda' \Big\}\overset{3}{\pi}\underline{f}^0 + \\ + L_2 e^{-\kappa^3\tau_0}\int_0^{\tau_0} e^{\kappa^3\lambda} I^3 e^{-\kappa^3\lambda}\int_0^{\lambda} e^{\kappa^3\lambda'} I^3 d\lambda\, d\lambda' \overset{3}{\pi}\underline{f}^0 \tag{10.15}$$

where spatial homogeneity has been used repeatedly. It is easy to see that the first term on the right hand side of (10.15) corresponds to (7.96), the second term to (7.97) and the third to (7.98). Therefore, the quantity $\underline{F}^2$ obtained from (10.15) is divergent. We have thus established the connection between the breakdown of our asymptotic analysis in the short range and in the weak coupling expansions.

Now, we want to give a more detailed mathematical description of our divergence. This is done in the next section. This section will be concluded by discussing rapidly but completely the dilute, weakly coupled gas directly from the Liouville equation.

B. Dilute, Weakly Coupled Gas

From the hierarchy (6.15) we obtain with the choice of parameters (6.19)

$$\frac{\partial \underline{F}^1}{\partial t} = \epsilon^2 L \underline{F}^2 \tag{10.16}$$

$$\frac{\partial \underline{F}^2}{\partial t} + K^2 \underline{F}^2 = \epsilon I^2 \underline{F}^2 + \epsilon^2 L_2 \underline{F}^3 \tag{10.17}$$

$$\frac{\partial \underline{F}^3}{\partial t} + K^3 \underline{F}^3 = \epsilon I^3 \underline{F}^3 + \epsilon^2 L_3 \underline{F}^4 \tag{10.18}$$

$$\frac{\partial \underline{F}^4}{\partial t} + K^4 \underline{F}^4 = \epsilon I^4 \underline{F}^4 + \epsilon^2 L_4 \underline{F}^5 \tag{10.19}$$

$$\frac{\partial \underline{F}^s}{\partial t} + \mathcal{K}^s \underline{F}^s = \epsilon I^s \underline{F}^s + \epsilon^2 L_\sim \underline{F}^6 \tag{10.20}$$

We consider the Bogolubov problem

$$\underline{f}^K \underset{\tau_0}{\sim} 0 \qquad K \neq 0$$

(i) <u>Zeroth-Order Theory</u>.

We readily obtain

$$\underline{f}^0(\tau_0) = \underline{f}^0 \tag{10.21}$$

and

$$\underline{F}^{s0}(\tau_0) = \overset{s}{\pi}\underline{f}^0 \tag{10.22}$$

(ii) <u>First-Order Theory</u>.

We have

$$\underline{f}^0(\tau_1) = \underline{f}^0 \tag{10.23}$$

$$\underline{f}^1(\tau_0) = \underline{f}^1 = 0 \tag{10.24}$$

Also, for the s-body distribution:

$$\underline{F}^{s1}(\tau_0) = \int_0^{\tau_0} e^{-\mathcal{K}^s \lambda} d\lambda \, I^s \overset{s}{\pi} \underline{f}^0 \tag{10.25}$$

$$\underset{\tau_0}{\sim} \zeta^*(i\mathcal{K}^s) I^s \overset{s}{\pi} \underline{f}^0$$

(iii) Second-Order Theory.

We find, for the one-body distribution

$$\underline{f}^{\circ}(\tau_2) = \underline{f}^{o} \tag{10.26}$$

and

$$\underline{f}^{2}(\tau_o) = \underline{f}^{2} = 0 \tag{10.27}$$

For the s-body distribution, we find:

$$\underline{F}^{s2}(\tau_o) = e^{-\kappa^s \tau_o} \int_0^{\tau_o} e^{\kappa^s \lambda} I^s \frac{1 - e^{-\kappa^s \lambda}}{\kappa^s} \overset{s}{I} d\lambda \cdot \overset{s}{\Pi} \underline{f}^{\circ} \tag{10.28}$$

$$\underset{\tau_o}{\approx} (\zeta^* I)^s (\zeta^* I)^s \overset{s}{\Pi} \underline{f}^{\circ}$$

which improves (10.25) by repeating the interaction process.

(iv) Third-Order Theory.

We find, for the single-body distribution,

$$\frac{\partial f^{\circ}}{\partial \tau_3} = L(\zeta^{*}I)_{12} f_1^{\circ} f_2^{\circ} \tag{10.29}$$

which corresponds to (7.70) and

$$\frac{\partial f^{3}}{\partial \tau_0} = -L\int_{\tau_0}^{\infty} e^{-\kappa^2\lambda}\,d\lambda\, I^2 f^{\circ} f^{\circ} \underset{\tau_0}{\approx} \frac{1}{\tau_0^2} \tag{10.30}$$

which corresponds to (7.73). Similarly, for the two-body distribution

$$\begin{aligned} E^{23} &\underset{\tau_0}{\approx} (\zeta^{*}I)(\zeta^{*}I)(\zeta^{*}I) f^{\circ} f^{\circ} + \\ &+ \zeta^{*}_{12}\left[L_{13}(\zeta^{*}I)_{23} + L_{23}(\zeta^{*}I)_{13}\right] f^{\circ} f^{\circ} f^{\circ} \end{aligned} \tag{10.31}$$

The second term on the right hand side of (10.31) corresponds to (7.92). Clearly the correspondence cannot be term by term to all orders.

For the three-body function we have:

$$\begin{aligned} E^{33} &\underset{\tau_0}{\approx} (\zeta^{*}I)^3(\zeta^{*}I)^3(\zeta^{*}I)^3 \overset{3}{\Pi} f^{\circ} + \\ &+ \zeta^{*3}\left[L_{14}(\zeta^{*}I)_{24} + L_{14}(\zeta^{*}I)_{34} + \right. \\ &+ L_{24}(\zeta^{*}I)_{14} + L_{24}(\zeta^{*}I)_{34} + \\ &\left. + L_{34}(\zeta^{*}I)_{14} + L_{34}(\zeta^{*}I)_{24}\right] \overset{4}{\Pi} f^{\circ} \end{aligned} \tag{10.32}$$

(v) Fourth-Order Theory.

We have, for the kinetic condition:

$$\frac{\partial \underline{f}^{\circ}}{\partial \tau_4} = L(\zeta^* I)(\zeta^* I)\underline{f}^{\circ}\underline{f}^{\circ} \tag{10.33}$$

while the two-body distribution satisfies

$$\frac{\partial \underline{F}^{24}}{\partial \tau_0} + \kappa^2 \underline{F}^{24} = I^2 \underline{F}^{23} + L_2 \underline{F}^{32} - \frac{\partial \underline{F}^{23}}{\partial \tau_1} - \frac{\partial \underline{F}^{22}}{\partial \tau_2} - \frac{\partial \underline{F}^{21}}{\partial \tau_3} - \frac{\partial \underline{F}^{20}}{\partial \tau_4} \tag{10.34}$$

but

$$\frac{\partial \underline{F}^{23}}{\partial \tau_1} = \frac{\partial \underline{F}^{22}}{\partial \tau_2} = 0 \tag{10.35}$$

and

$$\frac{\partial F^{21}}{\partial \tau_3} = \frac{1 - e^{-\kappa^2 \tau_0}}{\kappa^2} I^2 \left[L_{13}(\zeta^*)_{13} + L_{23}(\zeta^* I)_{23}\right] \overset{3}{\Pi} \underline{f}^{\circ} \tag{10.36}$$

Also,

$$\frac{\partial \underline{F}^{20}}{\partial \tau_4} = \left[L_{13}(\zeta^* I)_{13}(\zeta^* I)_{13} + L_{23}(\zeta^* I)_{23}(\zeta^* I)_{23}\right] \overset{3}{\Pi} \underline{f}^{\circ} \tag{10.37}$$

Therefore,

$$
\begin{aligned}
\underline{F}^{24}_{\tau_0} \sim \zeta^*_{12}\{ & I^2(\zeta^*I)(\zeta^*I)(\zeta^*I)\underline{f}^0\underline{f}^0 + \\
& + (I\zeta^*)_{12}\left[L_{13}(\zeta^*I)_{23} + L_{23}(\zeta^*I)_{13}\right]\overset{3}{\pi}\underline{f}^0 + \\
& + L_2(\zeta^*I)^3(\zeta^*I)^3\overset{3}{\pi}\underline{f}^6 - \\
& - (\zeta^*I)_{12}\left[L_{13}(\zeta^*I)_{13} + L_{23}(\zeta^*I)_{23}\right]\overset{3}{\pi}\underline{f}^0 - \\
& - \left[L_{13}(\zeta^*I)_{13}(\zeta^*I)_{13} + L_{23}(\zeta^*I)_{23}(\zeta^*I)_{23}\right]\overset{3}{\pi}\underline{f}^0
\end{aligned}
\qquad (10.38)
$$

It is immediately clear that the very singular term in which the $\zeta^*\zeta^*$ product appears does not cancel. In the next section, this persistent difficulty will become more transparent.

(vi) Fifth-Order Theory.

We can readily write the kinetic condition:

$$
\begin{aligned}
\frac{\partial \underline{f}^0}{\partial \tau_5} = L\zeta^*_{12}\{ & I^2(\zeta^*I)^2(\zeta^*I)^2(\zeta^*I)^2\underline{f}^0\underline{f}^0 + \\
& + (I\zeta^*)_{12}\left[L_{13}(\zeta^*I)_{23} + L_{23}(\zeta^*I)_{13}\right]\overset{3}{\pi}\underline{f}^0 + \\
& + L_2(\zeta^*I)^3(\zeta^*I)^3\overset{3}{\pi}\underline{f}^0 - \\
& - (\zeta^*I)_{12}\left[L_{13}(\zeta^*I)_{13} + L_{23}(\zeta^*I)_{23}\right]\overset{3}{\pi}\underline{f}^0 - \\
& - \left[L_{13}(\zeta^*I)_{13}(\zeta^*I)_{13} + L_{23}(\zeta^*I)_{23}(\zeta^*I)_{23}\right]\overset{3}{\pi}\underline{f}^0\}
\end{aligned}
\qquad (10.39)
$$

This formal result has clearly surpassed the limits of validity of our asymptotic expansion. In particular, one readily recognizes in the fourth term on the right hand side the contribution

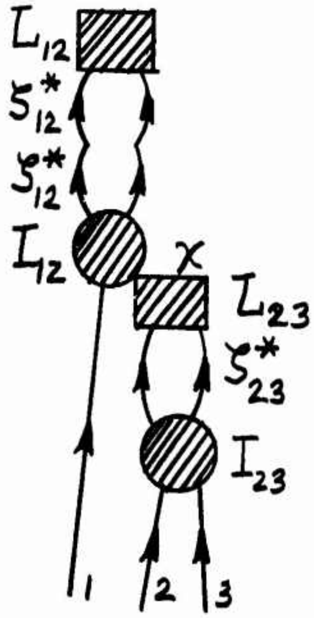


Fig. 17.

SECTION 11

THE CONFIGURATION SPACE

In this section we first discuss a decomposition of the configuration space of the four distribution functions F^1 to F^4 in terms of the interaction sphere of each of the particles. We then show how this decomposition classifies the kinematic degeneracies in the n-body collisions. This analysis will make clear that we have "over corrected" for secularities in the region of configuration space where our asymptotic expansion fails.

It is a pleasure to acknowledge the collaboration of Prof. W. Hayes for the work of this section.

A. Decomposition of the Configuration Space

We remind the reader that we have confined ourselves to spatially homogeneous gases.

(i) The one-body distribution function is completely independent of position. Therefore, the one-body configuration space C^1, is one point.

(ii) The two-body configuration space C^2 is the half line:

$$|\underline{x}_{12}| > 0 \tag{11.1}$$

We decompose C^2 as follows

$$C^2_+ : \qquad |\underline{x}_{12}| > r_0 \tag{11.2}$$

and

$$C^2_- : \qquad |\underline{x}_{12}| \leq r_0 \tag{11.3}$$

C^2_+ is the "outer" region, C^2_- the "inner".

(iii) The three-body confirmation space, C^3, is a non-euclidean three-dimensional manifold with a rather intricate topology. It decomposes into 8 regions:

$$C_1^3 : \ |\underline{x}_{12}| > r_0 , \quad |\underline{x}_{13}| > r_0 , \quad |\underline{x}_{23}| > r_0 \tag{11.4}$$

corresponding to configurations in which all three bodies are outside each other's interaction sphere,

$$\begin{aligned} C_2^3 &: \ |\underline{x}_{12}| < r_0 , \quad |\underline{x}_{13}| > r_0 , \quad |\underline{x}_{23}| > r_0 \\ C_3^3 &: \ |\underline{x}_{12}| > r_0 , \quad |\underline{x}_{13}| < r_0 , \quad |\underline{x}_{23}| > r_0 \\ C_4^3 &: \ |\underline{x}_{12}| > r_0 , \quad |\underline{x}_{13}| > r_0 , \quad |\underline{x}_{23}| < r_0 \end{aligned} \tag{11.5}$$

for which there is a pair interaction,

$$\begin{aligned} C_5^3 &: \ |\underline{x}_{12}| < r_0 , \quad |\underline{x}_{13}| < r_0 , \quad |\underline{x}_{23}| > r_0 \\ C_6^3 &: \ |\underline{x}_{12}| < r_0 , \quad |\underline{x}_{13}| > r_0 , \quad |\underline{x}_{23}| < r_0 \\ C_7^3 &: \ |\underline{x}_{12}| > r_0 , \quad |\underline{x}_{13}| < r_0 , \quad |\underline{x}_{23}| < r_0 \end{aligned} \tag{11.6}$$

Each of these regions corresponds to configurations in which two pairs are within each other's interaction spheres. Finally,

$$C_8^3 : \ |\underline{x}_{12}| < r_0 , \quad |\underline{x}_{13}| < r_0 , \quad |\underline{x}_{23}| < r_0 \tag{11.7}$$

for configurations in which all three pairs are interacting.

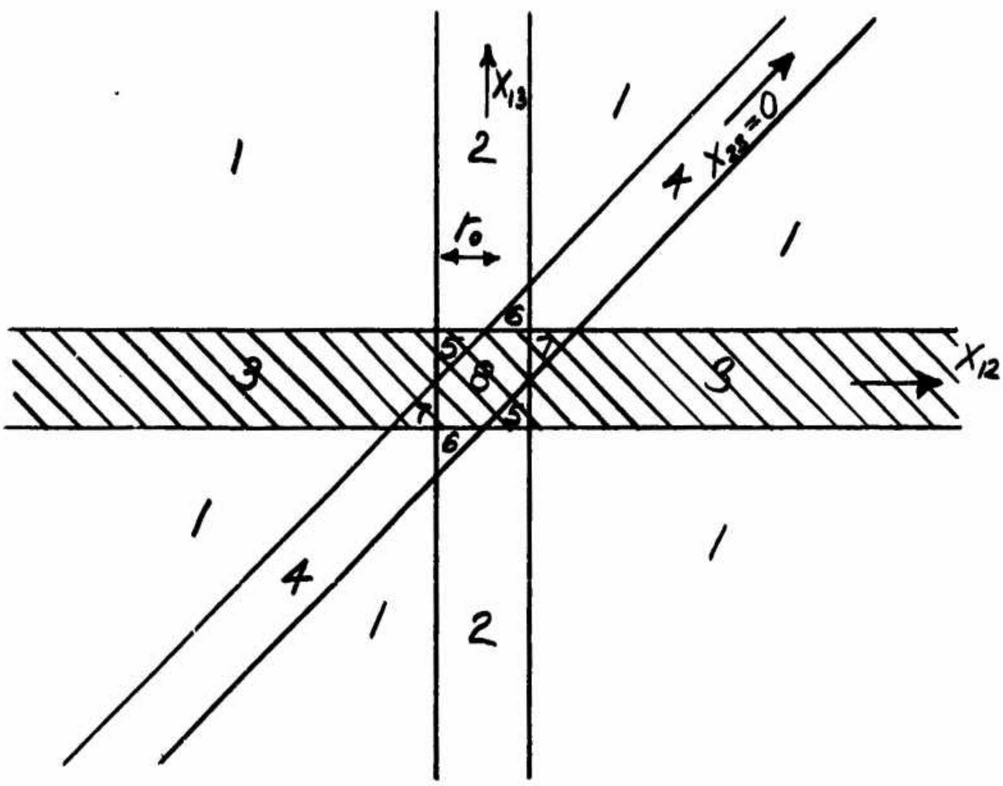
This decomposition is difficult to visualize except for one-dimensional gases. We treat therefore this case in detail. For the purpose of analyzing the divergence in F^2 due to three-body collisions, two integrals are of relevance

$$\Lambda_1 \equiv L_{13} \underline{F}^3$$

$$\Lambda_2 \equiv L_{23} \underline{F}^3 \qquad (11.8)$$

the L operators providing a cut off outside certain strips.

Thus, for the L_{13} integral we consider:



L_{13} integral

Fig. 18.

The C_i^3 are disconnected regions whence the rather intricate topology. Clearly only the hatched horizontal strip contributes to Λ_1 .

For the L_{23} integral, we consider instead:

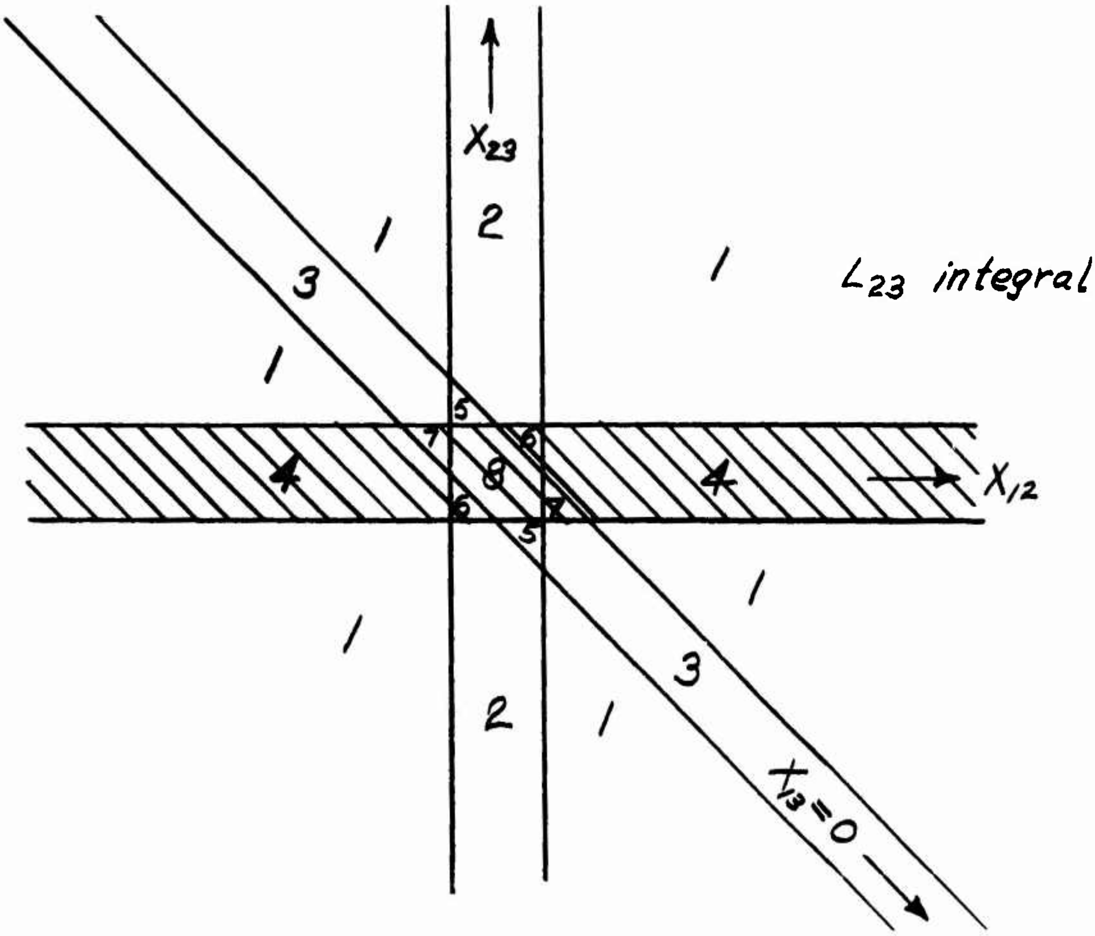


Fig. 19.

Clearly only the horizontal hatched strip contributes to Λ_2 .

The inner squares of the two diagrams are particularly important since they alone contribute to the rate of change of the one-body distribution, i.e. one has two successive cut-offs:

$$L_{12}\lambda L_{2} = L_{12}\lambda L_{13} + L_{12}\lambda L_{23} \tag{11.9}$$

If one did not correct the time derivative this contribution would lead to a triple product integral for the correction to Boltzmann's equation.

(iv) The four-body configuration space can be decomposed similarly. The system has six degrees of freedom which are conveniently chosen as

$$C^4: \quad |\underline{x}_{12}|, |\underline{x}_{13}|, |\underline{x}_{14}|, |\underline{x}_{23}|, |\underline{x}_{24}|, |\underline{x}_{34}| \tag{11.10}$$

That is, all six distances. The corresponding sixty-four regions are readily defined.

We have:

1 region with no pairs interacting
6 regions with one pair interacting
15 regions with two pairs interacting
20 regions with three pairs interacting
15 regions with four pairs interacting
6 regions with five pairs interacting
1 region with six pairs interacting

The four-body distribution is the highest for which it is possible to decompose the configuration space by means of the relative distances. The following types of integrals are those of relevance:

$$J^3 = [L_{14} + L_{24} + L_{34}] \underline{F}^4 \tag{11.11}$$

corresponding to $\partial \underline{F}^3 / \partial t$

$$J^2 = L_2 \lambda L_3 \underline{E}^4$$
$$= (L_{13}+L_{23}) \lambda (L_{14}+L_{24}+L_{34}) \underline{E}^4 \qquad (11.12)$$

corresponding to $\partial \underline{E}^2/\partial t$

$$J' = L \lambda' L_2 \lambda L_3 E^4$$
$$= L_{12} \lambda' (L_{13}+L_{23}) \lambda (L_{14}+L_{24}+L_{34}) \underline{E}^4 \qquad (11.13)$$

corresponding to $\partial \underline{E}'/\partial t$.

B. Kinematical Degeneracies

The decomposition of the configuration space is particularly useful because it allows for a complete classification of the n-body collisions in which only $m\,(m<n)$ of the bodies participate. In fact, the $\mathcal{H}^s$ operator undergoes definite contractions in the different regions of configuration space.

(i) Two-Body Degeneracies.

From (11.2) and (11.3) we have:

$$C_+^2 : \quad \mathcal{H}^2 \longrightarrow \mathcal{K}^2 \qquad (11.14)$$

and

$$C_-^2 : \quad \mathcal{H}^2 \longrightarrow \mathcal{K}^2 - \mathcal{I}^2 \qquad (11.15)$$

Schematically,

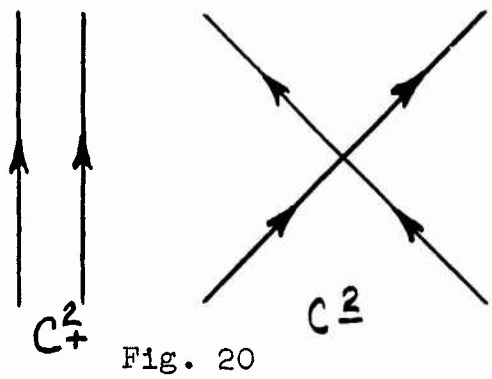


Fig. 20

Only the configurations belonging to C_- contribute to the collision integral.

(ii) Three-Body Degeneracies.

Referring to equations (11.4) to (11.7) we have

$$C_1^3 : \qquad \mathcal{H}^3 \longrightarrow \mathcal{K}^3 \tag{11.16}$$

in the region where no pairs interact. Also,

$$\begin{aligned} C_2^3 &: \qquad \mathcal{H}^3 \longrightarrow \mathcal{H}_{12} + \mathcal{K}_3 \\ C_3^3 &: \qquad \mathcal{H}^3 \longrightarrow \mathcal{H}_{13} + \mathcal{K}_2 \\ C_4^3 &: \qquad \mathcal{H}^3 \longrightarrow \mathcal{H}_{23} + \mathcal{K}_1 \end{aligned} \tag{11.17}$$

in the regions where only one pair interacts. Similarly,

$$\begin{aligned} C_5^3 &: \qquad \mathcal{H}^3 \longrightarrow \mathcal{K}^3 - I_{12} - I_{13} \\ C_6^3 &: \qquad \mathcal{H}^3 \longrightarrow \mathcal{K}^3 - I_{12} - I_{23} \\ C_7^3 &: \qquad \mathcal{H}^3 \longrightarrow \mathcal{K}^3 - I_{13} - I_{23} \end{aligned} \tag{11.18}$$

in the regions where two pairs interact. Finally in the region where all three pairs interact

$$C_8^3 : \qquad \mathcal{H}^3 \longrightarrow \mathcal{K}^3 - I_{12} - I_{23} - I_{31} \tag{11.19}$$

Schematically:

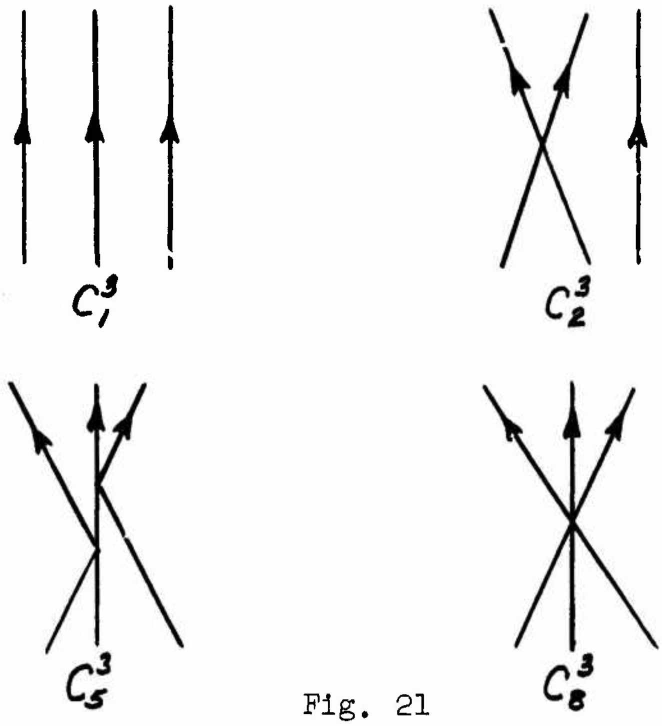


Fig. 21

The reduction of the Hamiltonian entails the existence of particular constants of the motion corresponding to the reduced number of bodies actually colliding. We turn now to a discussion of the effect of three-body collisions on the two-body function (13). We have, from (9.33)

$$\frac{\partial \underline{F}^{21}}{\partial \tau_0} + \mathcal{H}^2 \underline{F}^{21} =$$

$$= L_2 e^{-\mathcal{H}^3 \tau_0} \underline{f}^0 \underline{f}^0 \underline{f}^0 - e^{-\mathcal{H}_{12} \tau_0} \left[L_{13} S_{13} + L_{23} S_{23} \right] \underline{f} \underline{f} \underline{f}^0 \tag{11.20}$$

The second term on the right hand side of (11.20) is the correction to the time derivative. That is, it equals

$$\partial E^{22} / \partial \tau_1$$

and coincides asymptotically in τ_0 with the quantity

$$\int \frac{\delta F^{20}[F^1]}{\delta F^1(\underline{\xi})} A^1(\underline{\xi} \mid F^1) d\underline{\xi}$$

of Bogolubov's synchronized technique. It represents degenerate three-body collisions which are "successive" two-body interactions. This contribution to $\underline{F}^{21}$ grows linearly in τ_0 by construction. The time derivative has been in fact corrected precisely to absorb the growing contributions of the direct perturbation expansion. The linear growth of this term is in fact very readily proved directly. We have, rewriting slightly (11.20):

$$\begin{aligned} &\frac{\partial \underline{F}^{21}}{\partial \tau_0} + \mathcal{H}^2 \underline{F}^{21} = \\ &= L_2 \underline{F}^{30}(\tau_0) - e^{-\mathcal{H}^2 \tau_0} \frac{\partial}{\partial \tau_1} (\underline{f}^0 \underline{f}^0) \end{aligned} \tag{11.21}$$

whence

$$\begin{aligned} \underline{F}^{21}(\tau_0) &= e^{-\mathcal{H}^2 \tau_0} \int_0^{\tau_0} e^{\mathcal{H}^2 \lambda} L_2 \underline{F}^{30}(\lambda) d\lambda \\ &- e^{-\mathcal{H}^2 \tau_0} \tau_0 \frac{\partial}{\partial \tau_1} [\underline{f}^0(\underline{v}_1) \underline{f}^0(\underline{v}_2)] \end{aligned} \tag{11.22}$$

Therefore, the contribution to $\underline{E}^{21}$, $\delta\underline{E}^{21}$ from the second term on the right hand side is

$$\delta\underline{F}^{21} \underset{\tau_0}{\sim} -\tau_0 \frac{\partial}{\partial \tau_1}\left[\underline{f}^0(\underline{V}_1) f^0(\underline{V}_2)\right] \tag{11.23}$$

where, as in Section (9) the $\underline{V}_1$ denote the asymptotic velocities in the two-body collision.

In order that (11.20) give an $\epsilon\underline{E}^{21}$ that can be considered $O(\epsilon)$ with respect to $\underline{F}^{20}$, the "successive" two-body collisions must cancel against the direct three-body contribution $L_2\underline{F}^{20}$. The divergence discussed before in the short-range theory as well as in the weak coupling and dilute weak coupling expansions correspond to the fact that the desired cancellation in fact fails.

This important result is obtained by referring to formulae (11.16) to (11.19) for the reduction of the three-body Hamiltonian. We rewrite (11.20) in two different regions of C^2.

$|\underline{X}_{12}| > 2r_0$

$$\frac{\partial \underline{E}^{21}}{\partial \tau_0} + \mathcal{K}^2\underline{E}^{21} = \left[L_{13}e^{-\mathcal{H}_{13}\tau_0} + L_{23}e^{-\mathcal{H}_{23}\tau_0}\right]\overset{3}{\Pi}\underline{f}^0 - \tag{11.24}$$

$$-\left[L_{13}S_{13} + L_{23}S_{23}\right]\overset{3}{\Pi}\underline{f}^0 \qquad \underset{\tau_0}{\sim} 0$$

The cancellation of the secularity is therefore complete in the "outer arms". However, in the "inner" region $\underline{C}^2$

$|\underline{X}_{12}| < r_0$ $\qquad \dfrac{\partial \underline{E}^{21}}{\partial \tau_0} + \mathcal{H}^2\underline{F}^{21} =$

$$= \left\{L_{13}(C_5^3 \cup C_8^3) + L_{23}(C_6^3 \cup C_8^3)\right]e^{-\mathcal{H}^3\tau_0} - e^{-\mathcal{H}_{12}\tau_0}\left[L_{13}S_{13} + L_{23}S_{23}\right]\right\}\overset{3}{\Pi}\underline{f}^0 \tag{11.25}$$

there is clearly no cancellation. We used the notation $C_i^3 \cup C_j^3$ to denote the union of C_i^3 and C_j^3.

There is an intermediate region, $r_0 < |\underline{x}_{12}| < 2r_0$, in which the cancellation also fails to be complete.

The physical reason for the divergent contribution to the three-body collisions corresponds to the fact that successive two-body collisions have a very large (three-body) correlation length, of the order of λ (the mean free path), and give rise therefore to very persistent (in fact secular) contributions. It is clear however that the entire contribution from two-body collisions to the relaxation mechanism of the gas toward the equilibrium state is taken into account already (<u>and completely</u>) by the Boltzmann collision integral. If one is not wise enough not to count things twice, one is left with a very big result!

SECTION 12

DEBYE EXPANSION

It is convenient to transform the hierarchy (6.15):

$$\frac{\partial F^S}{\partial t} + \mathcal{H}^S F^S = (nr_0^3)\left(\frac{\phi_0}{MV_{th}^2}\right) L_S F^{SH} \tag{12.1}$$

into the equations for the correlation functions. We will be concerned with both the two-body correlation g and the three-body correlation h. Substituting the cluster expansion (8.2), (8.3), (8.4) into (12.1) we find after some calculation:

$$\frac{\partial F'}{\partial t} + K_1 F' = (nr_0^3)\left(\frac{\phi_0}{mV_{th}^2}\right) Lg \tag{12.2}$$

$$\frac{\partial g}{\partial t} + \mathcal{H}^2 g = \left(\frac{\phi_0}{mV_{th}^2}\right) I^2 F'F' + $$
$$+ (nr_0^3)\left(\frac{\phi_0}{mV_{th}^2}\right)\left\{L_{13} F_1 g_{23} + L_{23} F_2 g_{13} + L_2 h_{123}\right\} \tag{12.3}$$

and

$$\frac{\partial h}{\partial t} + \mathcal{H}^3 h = \left(\frac{\phi_0}{mV_{th}^2}\right)\left[I_{12}(F_1 g_{23} + F_2 g_{13}) + I_{13}(F_1 g_{23} + F_3 g_{12}) + I_{23}(F_2 g_{13} + F_3 g_{12})\right] +$$
$$+ (nr_0^3)\left(\frac{\phi_0}{mV_{th}^2}\right)\left\{L_{14}(g_{12} g_{34} + g_{13} g_{34}) + L_{24}(g_{12} g_{34} + g_{14} g_{23}) + \right. \tag{12.4}$$
$$+ L_{34}(g_{13} g_{24} + g_{14} g_{23}) + L_{14} F_1 h_{234} + L_{24} F_2 h_{134} + L_{34} F_3 h_{124} +$$
$$\left. + L_3 K_{1234}\right\}$$

where K is the four-body correlation function. If we now introduce the choice (6.18) for the basic parameters,

$$\frac{\phi_0}{m v_{th}^2} = \epsilon \, , \qquad n r_0^3 = \frac{1}{\epsilon} \tag{12.5}$$

and introduce the extension coordinates (7.29) together with

$$\begin{aligned} f^K &\Longrightarrow \underline{f}^K \\ g^K &\Longrightarrow \underline{g}^K \\ h^K &\Longrightarrow \underline{h}^K \end{aligned} \tag{12.6}$$

we find immediately that the equations do not decouple at all in lowest order. We introduce now the simplifying assumptions

$$\underline{g} = O(\epsilon) , \quad \underline{h} = O(\epsilon^2) , \quad \underline{K} = O(\epsilon^3) \tag{12.7}$$

This choice is in the spirit of the simple initial value problem. We find immediately for the one-body distribution:

$$\frac{\partial \underline{f}^0}{\partial \tau_0} = 0 \tag{12.8}$$

$$\frac{\partial \underline{f}^1}{\partial \tau_0} + \frac{\partial \underline{f}^0}{\partial \tau_1} = L \underline{g}^1 \tag{12.9}$$

$$\frac{\partial \underline{f}^2}{\partial \tau_0} + \frac{\partial \underline{f}^1}{\partial \tau_1} + \frac{\partial \underline{f}^0}{\partial \tau_2} = L \underline{g}^2 \tag{12.10}$$

The two-body correlation satisfies:

$$\frac{\partial \underline{g}^1}{\partial \tau_0} + \mathcal{K}^2 \underline{g}^1 = I^2 \underline{f}^0 \underline{f}^0 + L_{13} \underline{f}_1^0 \underline{g}_{23}^1 + L_{23} \underline{f}_2^0 \underline{g}_{13}^1 \tag{12.11}$$

and

$$\begin{aligned}\frac{\partial \underline{g}^2}{\partial \tau_0} + \mathcal{K}^2 \underline{g}^2 = {} & I^2(\underline{f}^1 \underline{f}^0 + \underline{f}^0 \underline{f}^1) - \frac{\partial \underline{g}^1}{\partial \tau_1} + I^2 \underline{g}^1 + \\ & + L_{13} \underline{f}_1^1 \underline{g}_{23}^1 + L_{23} \underline{f}_2^1 \underline{g}_{13}^1 + L_{13} \underline{f}_1^0 \underline{g}_{23}^2 + \\ & + L_{23} \underline{f}_2^0 \underline{g}_{13}^2 + (L_{13} + L_{23}) h_{123}^2\end{aligned} \tag{12.12}$$

The three-body correlation satisfies:

$$\begin{aligned}\frac{\partial \underline{h}^2}{\partial \tau_0} + \mathcal{K}^3 \underline{h}^2 = {} & I_{12}(\underline{f}_1^0 \underline{g}_{23}^1 + \underline{f}_2^0 \underline{g}_{13}^1) + I_{13}(\underline{f}_1^0 \underline{g}_{23}^1 + \underline{f}_3^0 \underline{g}_{12}^1) + I_{23}(\underline{f}_2^0 \underline{g}_{13}^1 + \underline{f}_3^0 \underline{g}_{12}^1) + \\ & + L_{14}(\underline{g}_{12}^1 \underline{g}_{34}^1 + \underline{g}_{13}^1 \underline{g}_{24}^1) + L_{24}(\underline{g}_{12}^1 \underline{g}_{34}^1 + \underline{g}_{14}^1 \underline{g}_{23}^1) + L_{34}(\underline{g}_{13}^1 \underline{g}_{24}^1 + \underline{g}_{14}^1 \underline{g}_{23}^1) + \\ & + L_{14} \underline{f}_1^0 \underline{h}_{234}^2 + L_{24} \underline{f}_2^0 \underline{h}_{134}^2 + L_{34} \underline{f}_3^0 \underline{h}_{124}\end{aligned} \tag{12.13}$$

From (12.8), we have immediately:

$$\underline{f}^0(\tau_0) = \underline{f}^0 \tag{12.14}$$

The equation (12.11) for $\underline{g^1}$ is an integrodifferential equation in which all the time dependence is in the quantity $\underline{g^1}$. We can rewrite it as:

$$\frac{\partial \underline{g}^1}{\partial \tau_0} + (\mathcal{K}^2 - \Gamma^2)\underline{g}^1 = I^2 \underline{f}^0 \underline{f}^0 \tag{12.15}$$

where

$$\Gamma^2[\underline{f}^\circ]\underline{g}^1 = L_{13}\underline{f}_1^\circ\underline{g}_{23}^1 + L_{23}\underline{f}_2^\circ\underline{g}_{13}^1 \tag{12.16}$$

is a linear operator on $\underline{g}^1$. The solution of (12.15) is

$$\underline{g}^1(\tau_0) = e^{-(K^2-\Gamma^2)\tau_0}\underline{g}^1(0) + \int_0^{\tau_0} e^{-(K^2-\Gamma^2)\lambda} d\lambda\, I^2\underline{f}^\circ\underline{f}^\circ \tag{12.17}$$

We shall be concerned with the simple initial value problem and therefore only the second term is of interest. We then have

$$\underline{g}^1 \underset{\tau_0}{\sim} \int_0^\infty e^{-(K^2-\Gamma^2)\lambda} d\lambda\, I^2\underline{f}^\circ\underline{f}^\circ \tag{12.18}$$

Before we proceed, we shall make the $\underline{f}^\circ$ dependence of the propagator more explicit. For this purpose, we consider now the homogeneous equation:

$$\frac{\partial G}{\partial \tau_0} + (K^2 - \Gamma^2)G = 0 \tag{12.19}$$

whose solution is

$$G(\tau_0) = e^{-[K^2-\Gamma^2]\tau_0}G(0) = e^{-D\tau_0}G(0) \tag{12.20}$$

The Debye operator D_{12} can be factorized as a product (14)

$$D_{12} = D_1 D_2 \tag{12.21}$$

corresponding to the product solutions of (12.19):

$$G_{12} = G_1 G_2$$

where

$$\mathcal{D}_j \cdot G_j = \mathcal{K}_j \cdot G_j - \mathcal{L}_{j3} f_j^{\circ} G_3 \qquad (12.22)$$

The G_j satisfy in fact:

$$\frac{\partial}{\partial \tau_0} G_j + \mathcal{D}_j \cdot G_j = 0 \qquad (12.23)$$

or, making explicit the $\mathcal{K}$ and $\mathcal{L}$ operators:

$$\frac{\partial}{\partial \tau_0} G_j + \underline{v}_j \cdot \underline{\nabla}_j G_j - \int d\underline{x}_3 \underline{\nabla}_j U_{j3} \cdot \underline{\nabla}_{v_j} \underline{f}_j^{\circ} \int d\underline{v}_3 G_3 = 0 \qquad (12.24)$$

Taking the Laplace transform with respect to τ_0 and Fourier transform with respect to $\underline{x}_j$:

$$p \gamma_j + i \underline{v}_j \cdot \underline{k}_j \gamma_j - \underline{\nabla}_{v_j} \underline{f}_j^{\circ} \cdot i k_j \tilde{U}(k) \int d\underline{v}_3 \gamma_3 = G_j(0) \qquad (12.25)$$

from which

$$\gamma_j = -\frac{1}{p + i \underline{v}_j \cdot \underline{k}_j} \nabla_{v_j} f_j^{\circ} \cdot i \underline{k}_j \tilde{U}(k) \int d\underline{v}_3 \gamma_3 + \frac{G_j(0)}{p + i \underline{v}_j \cdot \underline{k}_j} \qquad (12.26)$$

Integrating over $\underline{V}_j{}'$ we obtain an algebraic equation for $\int d\underline{v}_3 \gamma_3$

$$\int d\underline{v}_j \gamma_j = \int \frac{G_j(0)}{p + i\underline{v}_j \cdot \underline{K}_j} d\underline{v}_j - \int \frac{\nabla_{vj} \underline{f}_j^{\circ} \cdot i\underline{K}_j \tilde{U}}{p + i\underline{v}_j \cdot \underline{K}_j} d\underline{v}_j \int d\underline{v}_3 \gamma_3 \tag{12.27}$$

whence

$$\int d\underline{v}_3 \gamma_3 = \frac{\int \frac{G_j(0) \, d\underline{v}_j}{p + i\underline{V}_3 \cdot \underline{K}_j}}{1 + i\int d\underline{v}_3 \frac{\nabla_{v_3} \underline{f}_3^{\circ} \cdot i\underline{K}_j \tilde{U}}{p + i\underline{V}_3 \cdot \underline{K}_j}} \tag{12.28}$$

We substitute (12.28) into (12.26):

$$\gamma_j = \frac{G_j(0)}{p + i\underline{V}_j \cdot \underline{K}_j} -$$

$$- \frac{\nabla_{v_j} \underline{f}_j^{\circ} \cdot i\underline{K}_j \tilde{U}}{p + i\underline{V}_j \cdot \underline{K}_j} \; \frac{\int \frac{G_3(0) \, d\underline{V}_3}{p + i\underline{V}_3 \cdot \underline{K}_j}}{1 + i\int d\underline{v}_3 \frac{\nabla_{v_3} f_3^{\circ} \cdot i\underline{K}_j \cdot \tilde{U}}{p + i\underline{V}_3 \cdot \underline{K}_j}} \tag{12.29}$$

We have then

$$G_j(\tau_0) = \frac{1}{2\pi i}\int_C e^{p\tau_0}\,\gamma_j(p)\,dp \tag{12.30}$$

where, as usual, the contour C is parallel to the imaginary axis and to the right of the origin. The somewhat intricate P dependence of γ makes it difficult to see explicitly the τ_0 dependence of G_R .

We now insert (12.17) into (12.9):

$$\begin{aligned} \frac{\partial f^0}{\partial \tau_1} &= L\,\zeta^*(i[K^2-\Gamma^2])\, I^2 \underline{f}^0 \underline{f}^0 \\ &= L\,\frac{\zeta^*(iK^2)}{\underline{\epsilon}}\, I^2 \underline{f}^0 \underline{f}^0 \end{aligned} \tag{12.31}$$

where we have introduced the "dielectric constant" operator $\underline{\epsilon}$ by

$$\underline{\epsilon}\,\zeta^*(i[K^2-\Gamma^2]) \equiv \zeta^*(iK^2) \tag{12.32}$$

The form (12.31) of the kinetic equation suggests the connection with the Landau equation. This is established by a "polarization expansion" in powers of Γ^2 which is analogous to the "momentum transfer" expansion in powers of I^2.

To obtain the next approximation in the theory, we must solve the equation (12.13) for $\underline{h}^2$. It is clear first of all that there are product solutions of the type (12.23). We can write

$$\begin{aligned} \frac{\partial \underline{h}^2}{\partial \tau_0} + [K^3-\Gamma^3]\underline{h}^2 \approx \tau_0 \Big\{ & I_{12}[\zeta^*(i[K_{23}-\Gamma_{23}])I_{23} + (\zeta^* I)_{13}] + \\ & + I_{13}[(\zeta^* I)_{23} + (\zeta^* I)_{12}] + I_{23}[(\zeta^* I)_{13} + (\zeta^* I)_{12}]\Big\} \prod^{3} \underline{f}^0 \end{aligned} \tag{12.33}$$

$$+\left\{L_{14}\left[(\zeta^*I)_{12}(\zeta^*I)_{34}+(\zeta^*I)_{13}(\zeta^*I)_{24}\right]+\right.$$

$$+L_{24}\left[(\zeta^*I)_{12}(\zeta^*I)_{34}+(\zeta^*I)_{14}(\zeta^*I)_{23}\right]+$$

$$\left.+L_{34}\left[(\zeta^*I)_{13}(\zeta^*I)_{24}+(\zeta^*I)_{14}(\zeta^*I)_{23}\right]\right\}\overset{4}{\Pi}\underline{f}^{\circ}$$

where the ζ^*_{ij} operators denote here

$$\zeta^*_{ij}\equiv\zeta^*(i[K_{ij}-\Gamma_{ij}]) \tag{12.39}$$

and where

$$\Gamma^3\underline{h}^2_{/123}=L_{14}\underline{f}^{\circ}_1\underline{h}^2_{234}+L_{24}\underline{f}^{\circ}_2\underline{h}^2_{/134}+L_{34}\underline{f}^{\circ}_3\underline{h}^2_{/124} \tag{12.35}$$

We can therefore write:

$$\begin{aligned}\underline{h}^2\underset{\tau_0}{\sim}\;&\zeta^{*3}\left\{I_{12}\left[(\zeta^*I)_{23}+(\zeta^*I)_{13}\right]+I_{13}\left[(\zeta^*I)_{23}+(\zeta^*I)_{12}\right]+\right.\\&\left.+I_{23}\left[(\zeta^*I)_{13}+(\zeta^*I)_{12}\right]\right\}\overset{3}{\Pi}\underline{f}^{\circ}+\\&+\zeta^{*3}\left\{L_{14}\left[(\zeta^*I)_{12}(\zeta^*I)_{34}+(\zeta^*I)_{13}(\zeta^*I)_{24}\right]+\right.\\&+L_{24}\left[(\zeta^*I)_{12}(\zeta^*I)_{34}+(\zeta^*I)_{14}(\zeta^*I)_{23}\right]+\\&\left.+L_{34}\left[(\zeta^*I)_{13}(\zeta^*I)_{24}+(\zeta^*I)_{14}(\zeta^*I)_{23}\right]\right\}\overset{4}{\Pi}\underline{f}^{\circ}\end{aligned} \tag{12.36}$$

where

$$\zeta^{*3} \equiv \zeta^{*}(i[K^3-\Gamma^3]) \qquad (12.37)$$

From (12.12) we find for the second order two-body correlation function,

$$\begin{aligned}
&\frac{\partial g^2}{\partial \tau_0} + (K^2-\Gamma^2)g^2 \,\widetilde{\tau_0} \\
&\widetilde{\tau_0}\,(I\zeta^*)_{12}\, I_{12}\, \underline{f}^0 \underline{f}^0 - \zeta^*_{12} I_{12}\left[L_{13}(\zeta^* I)_{13} + L_{23}(\zeta^* I)_{23}\right] \overset{3}{\Pi} \underline{f}^0 + \\
&+ L_2 \zeta^{*3}\Big\{ I_{12}\left[(\zeta^* I)_{23} + (\zeta^* I)_{13}\right] + I_{13}\left[(\zeta^* I)_{23} + (\zeta^* I)_{12}\right] + \\
&\quad + I_{23}\left[(\zeta^* I)_{13} + (\zeta^* I)_{12}\right]\Big\} \overset{3}{\Pi} \underline{f}^0 + \\
&+ L_2 \zeta^{*3}\Big\{ L_{14}\left[(\zeta^* I)_{12}(\zeta^* I)_{34} + (\zeta^* I)_{13}(\zeta^* I)_{24}\right] + \\
&+ L_{24}\left[(\zeta^* I)_{12}(\zeta^* I)_{34} + (\zeta^* I)_{14}(\zeta^* I)_{23}\right] + L_{34}\left[(\zeta^* I)_{13}(\zeta^* I)_{24} + (\zeta^* I)_{14}(\zeta^* I)_{23}\right]\Big\} \overset{4}{\Pi} \underline{f}^0
\end{aligned} \qquad (12.38)$$

We recognize immediately in the second term on the right hand side of (12.38) the very singular contribution with two coincident

Formally, we have now, using (12.10):

$$\begin{aligned}
&\frac{\partial \underline{f}'^0}{\partial \tau_2} = L(\zeta^* I)_{12}(\zeta^* I)_{12}\, \underline{f}^0 \underline{f}^0 + \\
&- L\zeta^*_{12}\Big\{\Big\{(\zeta^* I)_{12}\left[L_{13}(\zeta^* I)_{13} + L_{23}(\zeta^* I)_{23}\right] \\
&+ L_2 \zeta^{*3}\Big\{ I_{12}\left[(\zeta^* I)_{23} + (\zeta^* I)_{13}\right] + I_{13}\left[(\zeta^* I)_{23} + (\zeta^* I)_{12} + \right. \\
&\quad + I_{23}\left[(\zeta^* I)_{13} + (\zeta^* I)_{12}\right]\Big\} \quad \Big\}\Big\} \overset{3}{\Pi} \underline{f}^0 +
\end{aligned} \qquad (12.39)$$

$$+ L\zeta^*_{12} L\zeta^{*3}\Big\{ L_{14}\big[(\zeta^* I)_{12}(\zeta^* I)_{34} + (\zeta^* I)_{13}(\zeta^* I)_{24}\big] +$$

$$+ L_{24}\big[(\zeta^* I)_{12}(\zeta^* I)_{34} + (\zeta^* I)_{14}(\zeta^* I)_{23}\big] +$$

$$+ L_{34}\big[(\zeta^* I)_{13}(\zeta^* I)_{24} + (\zeta^* I)_{14}(\zeta^* I)_{23}\big]\Big\} \prod^{4} \underline{f}^{0}$$

This equation represents the correction to the kinetic equation of Bogolubov and Lenard. We have separated the right hand side into two-, three-, and four-body contributions. The presence of this latter in second-order emphasizes the great difference that exists between the Debye expansion and either the short range or the weak coupling expansions. We have schematically:

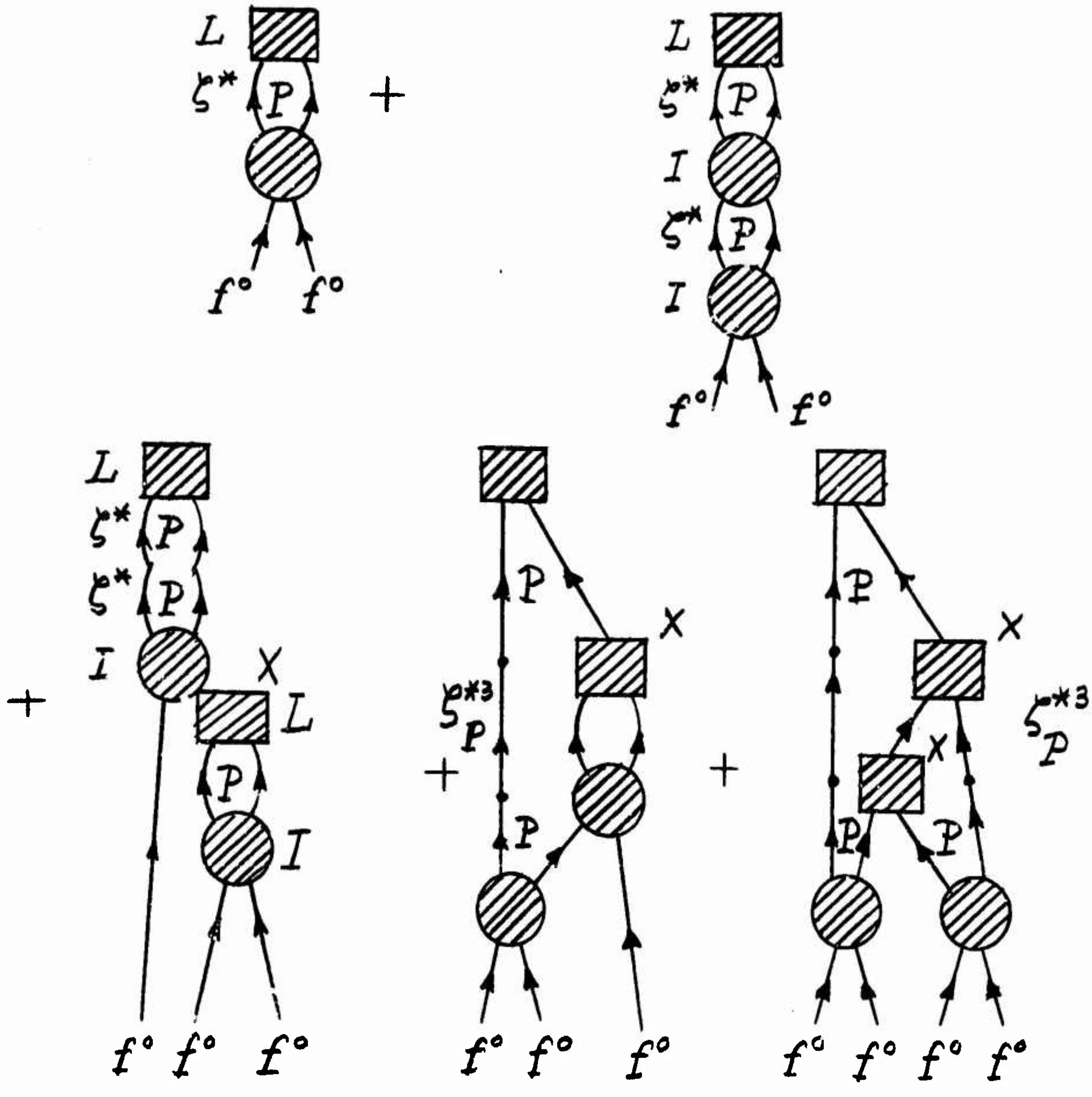

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Fig. 22.

For the Debye expansion, it is necessary to distinguish two types of complete initial value problems: the small correlation theory and the large correlation theory. The conditions for the former problem are of the somewhat intricate type

$$L e^{-(\chi^2-\Gamma^2)\tau_0} g'(0) \sim \frac{1}{\tau_0^{1+\eta}} \tag{12.40}$$

The latter problem is even more difficult. The zeroth-order hierarchy can be written as:

$$\frac{\partial \Psi}{\partial \tau_0} + \underline{K}\,\Psi = \underline{L}\,\Psi \tag{12.41}$$

where Ψ is a vector with components $\Psi_k \equiv F^{ko}$ and the $\underline{K}$ and $\underline{L}$ operators are appropriate matrices. The solution of (12.41) is

$$\Psi(\tau_0) = e^{-(\underline{K}-\underline{L})\tau_0}\,\Psi(0)$$

The kineticity conditions for this problem are therefore of the type

$$L\,\Psi_2(\tau_0) \underset{\tau_0}{\sim} \frac{1}{\tau_0^{1+\eta}} \tag{12.42}$$

Our argument for the breakdown of the kinetic expansion is now complete. We have found exactly the same divergent contribution in all four the regimes of interest.

SECTION 13

A GLOBAL THEORY

In this section we shall be conc[illegible]d with the Liouville equation itself. We write, for a spatially homogeneous gas

$$\frac{\partial F}{\partial t} + \left[K - \left(\frac{\phi_0}{m v_{th}^2}\right) I\right] F = 0 \tag{13.1}$$

and

$$\frac{\partial F'}{\partial t} = (n r_0^3)\left(\frac{\phi_0}{m v_{th}^2}\right) L_n F \tag{13.2}$$

where F is identical to F^N and

$$L_n = L \int \frac{d\underline{x}_2}{V} d\underline{v}_3 \ldots \frac{d\underline{x}_N}{V} d\underline{v}_N \tag{13.3}$$

We want now to consider (13.1) and (13.2) as directly coupled, i.e. we bypass the BBGKY hierarchy. We use the same extension technique that we have used so far.

A. Weak Coupling Theory (15)

Our perturbation equations for the N-body distribution function are:

$$\frac{\partial \underline{F}^0}{\partial \tau_0} + K\underline{F}^0 = 0 \tag{13.4}$$

$$\frac{\partial \underline{F}'}{\partial \tau_0} + K\underline{F}' = I\underline{F}^0 - \frac{\partial \underline{F}^0}{\partial \tau_1} \tag{13.5}$$

and for the one-body distribution function

$$\frac{\partial \underline{f}^{\circ}}{\partial \tau_0} = 0 \tag{13.6}$$

$$\frac{\partial \underline{f}^{1}}{\partial \tau_0} + \frac{\partial \underline{f}^{\circ}}{\partial \tau_1} = L_{\Gamma} \underline{F}^{\circ} \tag{13.7}$$

$$\frac{\partial \underline{f}^{2}}{\partial \tau_0} + \frac{\partial \underline{f}^{1}}{\partial \tau_1} + \frac{\partial \underline{f}^{\circ}}{\partial \tau_2} = L_{\Gamma} \underline{F}^{1} \tag{13.8}$$

We confine ourselves to the simple initial value problem. The zeroth-order equations yield immediately:

$$\underline{f}^{\circ}(\tau_0) = \underline{f}^{\circ} \tag{13.9}$$

and

$$\underline{F}^{\circ}(\tau_0) = \prod^{N} \underline{f}^{\circ} \tag{13.10}$$

The first-order equations are also readily obtained by noting that:

$$L_{\Gamma} \prod^{N} \underline{f}^{\circ} = 0 \tag{13.11}$$

because of the spatial homogeneity of the gas. We have therefore:

$$\underline{f}^{\circ}(\tau_1) = \underline{f}^{\circ} \tag{13.12}$$

and

$$\underline{f}^{1} = 0 \tag{13.13}$$

Also, from (13.5)

$$\underline{F}'(\tau_0) = \frac{1-e^{-\mathcal{K}\tau_0}}{\mathcal{K}} I \overset{N}{\Pi} \underline{f}^0 \underset{\tau_0}{\approx} \zeta^*(i\mathcal{K}) I \overset{N}{\pi} \underline{f}^0 \tag{13.14}$$

Substituting this result into the second-order equation (13.8)

$$\begin{aligned} \frac{\partial \underline{f}^0}{\partial \tau_2} &= L_r \zeta^*(i\mathcal{K}) I \overset{N}{\Pi} \underline{f}^0 \\ &= L \zeta^*_{12} I_{12} \underline{f}^0 \underline{f}^0 + (n r_0^3) \Lambda \end{aligned} \tag{13.15}$$

It is easy to verify, by means of a straightforward calculation that the coefficient of $(n r_0^3)$ is in fact identically zero. This result establishes the Landau equation as a direct consequence of the Liouville equation. A similar calculation establishes the next kinetic term. The following order diverges.

B. Short Range Expansion

We have again $$\frac{\partial \underline{f}^0}{\partial \tau_0} = 0 \tag{13.16}$$

For $\underline{F}^0$ we obtain, for the simple initial value problem:

$$\underline{F}^0(\tau_0) = e^{-\mathcal{H}\tau_0} \overset{N}{\Pi} \underline{f}^0 \tag{13.17}$$

From (13.2):

$$\frac{\partial \underline{f}^1}{\partial \tau_0} + \frac{\partial \underline{f}^0}{\partial \tau_1} = L_r e^{-\mathcal{H}\tau_0} \overset{N}{\Pi} \underline{f}^0 \underset{\tau_0}{\approx} L_r S \overset{N}{\Pi} \underline{f}^0 \tag{13.18}$$

whence

$$\frac{\partial \underline{f}^{\circ}}{\partial \tau_1} = L S^2 \underline{f}\underline{f}^{\circ} + O(n r_0^3) \tag{13.19}$$

Since the last term on the right hand side is negligible to this order of approximation, (13.19) establishes the Boltzmann equation directly from the Liouville equation.

We note that from (13.1)

$$\frac{\partial \underline{F}'}{\partial \tau_0} + \mathcal{H}\underline{F}^1 = -\frac{\partial \underline{F}^{\circ}}{\partial \tau_1} \tag{13.20}$$

Since the left hand side is Liouville's equation, we are now correcting our very first principles! From (13.17) it is clear that

$$\underline{F}' \sim \tau_0 \cdot \mathrm{const} \tag{13.21}$$

The arguments of the constant are the asymptotic values of the momentum in an N-body collision. We have clearly gone beyond the limit of validity of our asymptotic expansion.

C. Master Equations

We introduce

$$\phi \equiv \int F \frac{d\underline{x}_1}{V} \frac{d\underline{x}_2}{V} \cdots \frac{d\underline{x}_N}{V} \tag{13.22}$$

We readily obtain, from (13.1)

$$\begin{aligned} \frac{\partial \phi}{\partial t} &= \left(\frac{\phi_0}{m v_{th}^2}\right)\left(\frac{r_0^3}{V}\right) \int I F \frac{d\underline{x}_1}{V} \cdots \frac{d\underline{x}_N}{V} \\ &\equiv \left(\frac{\phi_0}{m v_{th}^2}\right)\left(\frac{r_0^3}{V}\right) L_c F \end{aligned} \tag{13.23}$$

We now consider (13.23) coupled to (13.1).

(i) Weakly coupled gas. We choose

$$\frac{\phi_0}{m v_{th}^2} = \epsilon << 1 , \quad \frac{r_0^3}{V} \simeq 1 \tag{13.24}$$

With the usual extension technique:

$$F^K \Rightarrow \underline{F}^K , \quad \phi^K \Longrightarrow \underline{\phi}^K \tag{13.25}$$

the Liouville equation (13.1) yields:

$$\frac{\partial \underline{F}^0}{\partial \tau_0} + K \underline{F}^0 = 0 \tag{13.26}$$

$$\frac{\partial \underline{F}^1}{\partial \tau_0} + K \underline{F}^1 = I \underline{F}^0 - \frac{\partial \underline{F}^0}{\partial \tau_1} \tag{13.27}$$

$$\frac{\partial \underline{F}^2}{\partial \tau_0} + K \underline{F}^2 = I \underline{F}^1 - \frac{\partial \underline{F}^1}{\partial \tau_1} - \frac{\partial \underline{F}^0}{\partial \tau_2} \tag{13.28}$$

From (13.23) on the other hand we obtain

$$\frac{\partial \underline{\phi}^0}{\partial \tau_0} = 0 \tag{13.29}$$

$$\frac{\partial \underline{\phi}^1}{\partial \tau_0} + \frac{\partial \underline{\phi}^0}{\partial \tau_1} = L_c \underline{F}^0 \tag{13.30}$$

$$\frac{\partial \underline{\phi}^2}{\partial \tau_0} + \frac{\partial \underline{\phi}^1}{\partial \tau_1} + \frac{\partial \underline{\phi}^0}{\partial \tau_2} = L_c \underline{F}^1 \tag{13.31}$$

$$\frac{\partial \underline{\phi}^3}{\partial \tau_0} + \frac{\partial \underline{\phi}^2}{\partial \tau_1} + \frac{\partial \underline{\phi}^1}{\partial \tau_2} + \frac{\partial \underline{\phi}^0}{\partial \tau_3} = L_c \underline{F}^2 \tag{13.32}$$

The analog of the simple initial value problem for the master equation is the "uniform" initial value problem by which we mean that the system of N-bodies is assumed to be initially in a state

$$\underline{F} = \underline{\phi} \cdot K \tag{13.33}$$

such that positions and momenta are statistically independent and furthermore such that the configuration space density is perfectly uniform. By normalization

$$K = 1 \tag{13.34}$$

Furthermore, we assume

$$\frac{\partial \underline{\phi}(0)}{\partial \epsilon} = 0 \tag{13.35}$$

Since (13.29) gives

$$\underline{\phi}^0(\tau_0) = \underline{\phi}^0 \tag{13.36}$$

we find, from (13.26)

$$\underline{F}^0(\tau_0) = e^{-\kappa \tau_0} \underline{\phi}^0 = \underline{\phi}^0 \tag{13.37}$$

Substituting this result into (13.30) we find

$$\frac{\partial \underline{\phi}'}{\partial \tau_0} + \frac{\partial \underline{\phi}^0}{\partial \tau_1} = L_c \underline{\phi}^0 \tag{13.38}$$

But one readily verifies that

$$L_c \underline{\phi}^0 = 0 \tag{13.39}$$

Whence, from (13.30)

$$\frac{\partial \underline{\phi}'}{\partial \tau_0} + \frac{\partial \underline{\phi}^0}{\partial \tau_1} = 0 \tag{13.40}$$

$$\underline{\phi}^0(\tau_1) = \underline{\phi}^0 \tag{13.41}$$

and

$$\underline{\phi}' = 0 \tag{13.42}$$

Substitution into (13.27) gives

$$\frac{\partial \underline{F}'}{\partial \tau_0} + K\underline{F}' = I\underline{\phi}^0 \tag{13.43}$$

in virtue of (13.41). We readily solve (13.43) to obtain

$$\underline{F}'(\tau_0) = \frac{1 - e^{-K\tau_0}}{K} I\underline{\phi}^0 \tag{13.44}$$

where we used (13.35) to obtain

$$\underline{F}'(0) = 0 \tag{13.45}$$

Substituting (13.44) into (13.31)

$$\frac{\partial \underline{\phi}^2}{\partial \tau_0} + \frac{\partial \underline{\phi}^0}{\partial \tau_2} = L_c \int_0^{\tau_0} e^{-\mathcal{K}\lambda} d\lambda \, I \underline{\phi}^0$$

$$\underset{\tau_0}{\sim} L_c \zeta^*(i\mathcal{K}) I \underline{\phi}^0 \tag{13.46}$$

The requirement that the approximation be uniform yields immediately

$$\frac{\partial \underline{\phi}^0}{\partial \tau_2} = L_c \zeta^*(i\mathcal{K}) I \underline{\phi}^0 \tag{13.47}$$

as well as the transient equation:

$$\frac{\partial \underline{\phi}^2}{\partial \tau_0} = -L_c \int_{\tau_0}^{\infty} e^{-\mathcal{K}\lambda} d\lambda \, I \underline{\phi}^0 \tag{13.48}$$

Equation (13.47) is a generalized master equation for a weakly coupled gas.

We now continue the expansion by using (13.28)

$$\frac{\partial \underline{F}^2}{\partial \tau_0} + \mathcal{K} \underline{F}^2 = I \, \frac{1 - e^{-\mathcal{K}\tau_0}}{\mathcal{K}} \, I \underline{\phi}^0 - L_c \zeta^*(i\mathcal{K}) I \underline{\phi}^0$$

$$\underset{\tau_0}{\sim} I \zeta^* I \, \underline{\phi}^0 - L_c \, \zeta^* I \underline{\phi}^0 \tag{13.49}$$

where we have used

$$\frac{\partial \underline{F}'}{\partial \tau_1} = 0 \tag{13.50}$$

and the master equation (13.47). From (13.49), formally:

$$\underline{F}^2 \sim \zeta^*(i\mathcal{K})[I(\zeta^* I) - L_c(\zeta^* I)]\underline{\phi}^0 \tag{13.51}$$

whence from (13.32):

$$\frac{\partial \underline{\phi}^0}{\partial \tau_3} = L_c \zeta^* [I(\zeta^* I) - L_c(\zeta^* \underline{I})]\underline{\phi}^0 \tag{13.52}$$

This "correction" to the master equation is in fact divergent just as (13.51) is divergent. The infinite term is the one in which the ζ^* operates on $L_c \Lambda$ which is a spatially homogeneous quantity.

Once more the master equation (13.37) in the asymptotic limit of $\partial\phi/\partial t$ while the next term cannot be obtained by continuing the expansion.

It is of considerable interest to consider initial distributions which are not "uniform". We thus introduce the analog of the complete initial value problem by defining the momentum-coordinates correlation ω :

$$F \equiv \phi + \omega \tag{13.53}$$

We now find to zeroth order:

$$\underline{\phi}^0(\tau_0) = \underline{\phi}^0 \tag{13.54}$$

but instead of (13.57):

$$\underline{E}^{0}(\tau_0) = \underline{\phi}^{0} + e^{-\varkappa\tau_0}\,\underline{\omega}^{0}(0) \tag{13.55}$$

The first-order equation (13.30) becomes therefore:

$$\frac{\partial \underline{\phi}^{1}}{\partial \tau_0} + \frac{\partial \underline{\phi}^{0}}{\partial \tau_1} = L_c\, e^{-\varkappa\tau_0}\,\underline{\omega}^{0}(0) \tag{13.56}$$

In order that the initial statistical interdependence between momenta and coordinates be completely forgotten by the momentum distribution $\underline{\phi}^{0}$ on the τ_1 scale, we must have

$$L_c\, e^{-\varkappa\tau_0}\,\underline{\omega}^{0}(0) \underset{\tau_0}{\approx} \frac{1}{\tau_0^{1+\eta}} \quad , \quad \eta > 0 \tag{13.57}$$

We can then conclude for (13.40).

From (13.27) we now obtain

$$\frac{\partial \underline{E}^{1}}{\partial \tau_0} + \varkappa \underline{E}^{1} = I\underline{\phi}^{0} + I e^{-\varkappa\tau_0}\,\underline{\omega}^{0}(0) - e^{-\varkappa\tau_0}\frac{\partial \underline{\omega}^{0}}{\partial \tau_1} \tag{13.58}$$

Substituting this result into (13.31):

$$\frac{\partial \underline{\phi}^2}{\partial \tau_0} + \frac{\partial \underline{\phi}^0}{\partial \tau_2} = L_c \int_0^{\tau_0} e^{-\kappa\lambda} d\lambda \, I \underline{\phi}^0 +$$

$$+ L_c e^{-\kappa\tau_0} \int_0^{\tau_0} e^{\kappa\lambda} \left[I e^{-\kappa\lambda} \underline{\omega}^0(0) - e^{-\kappa\lambda} \frac{\partial \underline{\omega}^0(0)}{\partial \tau_1} \right] d\lambda \qquad (13.59)$$

The condition of validity of our master equation (13.47) is therefore that the second term on the right hand side of (13.59) should decay for large τ_0 as $\frac{1}{\tau_0^{1+\eta}}$, $\eta > 0$.

(ii) Dilute Gas.

We now assume:

$$\frac{\phi_0}{m v_*^2} \simeq 1 , \quad \frac{r_0^3}{V} = \epsilon \qquad (13.60)$$

From (13.1) we have

$$\frac{\partial \underline{F}^0}{\partial \tau_0} + \mathcal{H} \underline{F}^0 = 0 \qquad (13.61)$$

$$\frac{\partial \underline{F}'}{\partial \tau_0} + \mathcal{H} \underline{F}' = - \frac{\partial \underline{F}^0}{\partial \tau_1} \qquad (13.62)$$

and from (13.23)

$$\frac{\partial \underline{\phi}^0}{\partial \tau_0} = 0 \qquad (13.63)$$

$$\frac{\partial \underline{\phi}^1}{\partial \tau_0} + \frac{\partial \underline{\phi}^0}{\partial \tau_1} = L_c \underline{F}^0 \tag{13.64}$$

$$\frac{\partial \underline{\phi}^2}{\partial \tau_0} + \frac{\partial \underline{\phi}^1}{\partial \tau_1} + \frac{\partial \underline{\phi}^0}{\partial \tau_2} = L_c \underline{F}^1 \tag{13.65}$$

The zeroth-order theory yields

$$\underline{\phi}^0(\tau_0) = \underline{\phi}^0 \tag{13.66}$$

and

$$\underline{F}^0(\tau_0) = e^{-\mathcal{H}\tau_0}\underline{\phi}^0 \underset{\tau_0}{\approx} S\underline{\phi}^0 \tag{13.67}$$

Substituting this result into (13.64):

$$\frac{\partial \underline{\phi}^1}{\partial \tau_0} + \frac{\partial \underline{\phi}^0}{\partial \tau_1} = L e^{-\mathcal{H}\tau_0}\underline{\phi}^0 \underset{\tau_0}{\approx} L_c S \underline{\phi}^0 \tag{13.68}$$

Therefore,

$$\frac{\partial \underline{\phi}^0}{\partial \tau_1} = L_c S \underline{\phi}^0 \tag{13.69}$$

This is the master equation for a dilute system (16). It is well to stress that "dilution" has different meanings here and in the short-range kinetic theory.

From (13.62)

$$\frac{\partial \underline{F}'}{\partial \tau_0} + \mathcal{H}\underline{F}' = -e^{-\mathcal{H}\tau_0} L_c S \underline{\phi}^0 \tag{13.70}$$

Therefore, formally:

$$\underline{F}'(\tau_0) = -\tau_0 e^{-\mathcal{H}\tau_0} L_c S \underline{\phi}^0 \underset{\tau_0}{\widetilde{}} -\tau_0 S L_c S \underline{\phi}^0 \tag{13.71}$$

Substitution into (13.65) would lead to a finite result if the clock were modified. This, however, would be a purely formal device in view of (13.71) itself. The generalized master equation (13.69) can be transformed by means of the identity:

$$KS = IS \tag{13.72}$$

Therefore

$$\begin{aligned} \frac{\partial \underline{\phi}^0}{\partial \tau_1} &= \int I S \underline{\phi}^0 \, d\underline{x}_1 \frac{d\underline{x}_2}{V} \cdots \frac{d\underline{x}_N}{V} \\ &= \sum^{N} \int \underline{V}_i \cdot \underline{\nabla}_i S \underline{\phi}^0 \, d\underline{x}_1 \frac{d\underline{x}_2}{V} \cdots \frac{d\underline{x}_N}{V} \\ &= \frac{1}{N-1} \sum_{i<j} \int \underline{V}_{ij} \cdot \underline{\nabla}_{ij} S \underline{\phi}^0 \, d\underline{x}_1 \frac{d\underline{x}_2}{V} \cdots \frac{d\underline{x}_N}{V} \end{aligned} \tag{13.73}$$

where use has been made of the spatial homogeneity of $\underline{\phi}^0$. The more familiar form of the master equation corresponds to the two-body collision contribution to (13.73).

SECTION 14

THEORY OF THREE-BODY COLLISIONS

We have seen that the asymptotic expansion of the Liouville equation that leads to the kinetic equations is insufficient to give a correct theory of three-body effects.

We retraced the difficulty to the fact that our expansions, that hinge on the one-body distribution function, do not count properly successive two-body collisions. Our attention was focused on the one-body distribution because our main interest was the kinetic regime of a gas, i.e. a regime in which a typical particle (or alternatively the average behavior of one particle) is sufficient to characterize the gas. We have also shown how we could "close" the kinetic equations by explciting the freedom of choice of solutions offered by our method of extension. Thus, our kinetic theory is closed and complete.

We now put forward the thesis that a correct understanding of the three-body effects demands a <u>different</u> expansion of the Liouville equation; one that uses as its fundamental stochastic variable not the one-body distribution function, but rather the two-body distribution.

The simple initial value problem for the kinetic theory hinged on molecular chaos. For the three-body collisions theory one needs a more informative initial condition. This is provided by Kirkwood's "hypothesis"

$$F^{3}_{\text{Kirkwood}} = \frac{F_{12}\, F_{23}\, F_{31}}{F_1\, F_2\, F_3} \tag{14.1}$$

The calculations can be carried out by the methods discussed in our study of the kinetic regime or in the synchronization language of Bogolubov by assuming synchronization to F^2. One finds for example, for short range forces,

$$\frac{\partial F^2}{\partial t} + \mathcal{H}^2 F^2 = L_2 S^3 \frac{\Pi S^2_{+\infty} F^2}{\Pi F^1} \tag{14.2}$$

The conditions for the validity of (14.2) are readily written in terms of

$$G^3 = F^3 - F^3_{\text{Kirkwood}} \tag{14.3}$$

in a manner analogous to that of the complete initial value problem of the kinetic theory. One readily verifies that the equation for F^1 implied by (14.2) gives the Boltzmann equation when molecular chaos is inserted in (14.1).

This theory contains one unsatisfactory aspect: It has not been possible as yet to construct a monotonic function.

The next higher level of approximation to the Liouville equation contains a chaos condition analogous to (14.1) that relates F^4 to F^3. Prof. W. Hayes has shown that the normalization of any such condition must depend on the state of the gas.

SECTION 15

INHOMOGENEOUS GAS AND HYDRODYNAMICS

Inhomogeneities are of fundamental interest in the theory of gases.

Their first appearance is with the bulk limit which is treated properly only by means of a wall potential. Clearly, the presence of external fields other than the walls induce inhomogeneities which are of considerable interest. A first approach to this problem has been given with the collaboration of Dr. J. McCune (17).

Lastly, inhomogeneities which are simply given are very important. We shall confine ourselves here with two remarks on this situation.

A. Kinetic Equations for Inhomogeneous Gases

We consider only the weakly coupled gas. The perturbation equations are:

$$\frac{\partial \underline{f}^0}{\partial \tau_0} + K' \underline{f}^0 = 0 \tag{15.1}$$

$$\frac{\partial \underline{f}^1}{\partial \tau_0} + K' \underline{f}^1 + \frac{\partial \underline{f}^0}{\partial \tau_1} = L \underline{F}^{20} \tag{15.2}$$

$$\frac{\partial \underline{f}^2}{\partial \tau_0} + K' \underline{f}^2 + \frac{\partial \underline{f}^1}{\partial \tau_1} + \frac{\partial \underline{f}^0}{\partial \tau_2} = L \underline{F}^{21} \tag{15.3}$$

for the one-body distribution,

$$\frac{\partial \underline{F}^{20}}{\partial \tau_0} + K^2 \underline{F}^{20} = 0 \tag{15.4}$$

$$\frac{\partial \underline{F}^{21}}{\partial \tau_0} + \mathcal{K}^2 \underline{F}^{21} + \frac{\partial \underline{F}^{20}}{\partial \tau_1} = L_2 \underline{F}^{30} \tag{15.5}$$

for the two-body distribution, and

$$\frac{\partial \underline{F}^{30}}{\partial \tau_0} + \mathcal{K}^3 \underline{F}^{30} = 0 \tag{15.6}$$

for the three-body distribution.

The zeroth-order theory is very simple for the simply initial problem. Thus,

$$\underline{f}^0(\tau_0) = e^{-\mathcal{K}_1 \tau_0} \underline{f}^0(0) \tag{15.7}$$

$$\underline{F}^{S0}(\tau_0) = \overset{S}{\pi}\, \underline{f}^0(\tau_0) \tag{15.8}$$

To first order we have

$$\frac{\partial \underline{f}^1}{\partial \tau_0} + \mathcal{K}_1 \underline{f}^1 + \frac{\partial \underline{f}^0}{\partial \tau_1} = L \underline{f}^0(\tau_0, \tau_1) \underline{f}^0(\tau_0, \tau_1) \tag{15.9}$$

a "solution" of this equation is the Vlasov condition:

$$\frac{\partial \underline{f}^0}{\partial \tau_1} = L \underline{f}^0(\tau_0, \tau_1) \underline{f}^0(\tau_0, \tau_1) \tag{15.10}$$

The two-body function satisfies (15.5) which reduces readily to:

$$\frac{\partial \underline{F}^{21}}{\partial \tau_0} + \kappa^2 \underline{F}^{21} = I_{12} \underline{f}^{\circ} \underline{f}^{\circ} \tag{15.11}$$

We now prove that the synchronized function:

$$\underline{F}^{21}[\underline{f}^{0}(\tau_0)] = \left\{ \zeta_{12}^{*} I_{12} \right\} \underline{f}^{\circ}(\tau_0) \underline{f}^{\circ}(\tau_0) \tag{15.12}$$

is a solution of Bogolubov's equation:

$$\int \frac{\delta \underline{F}^{21}}{\delta \underline{f}^{\circ}} A^{0} d \underline{\xi} + \kappa^2 \underline{F}^{21}[\underline{f}^{\circ}] = I^{2} \underline{F}^{20}[\underline{f}^{\circ}] \tag{15.13}$$

Substitution of (15.7) into (15.8) leads to:

$$\left\{ -(\zeta^{*} I) \kappa^2 + \kappa^2 (\zeta^{*} I) \right\} \underline{f}^{0} \underline{f}^{0} = I^{2} \underline{f}^{0} \underline{f}^{\circ} \tag{15.14}$$

which is an identity since one has:

$$\kappa^2 \left(\frac{P}{\kappa^2} + \pi \, \delta(i \kappa^2) \right) = 1 \tag{15.15}$$

We now obtain from (15.6) a generalization of (15.7). The general solution of (15.6) is in fact

$$E^{21}(\tau_0) = e^{-\kappa^2 \tau_0} \left\{ \int_0^{\tau_0} e^{+\kappa^2 \lambda} I^2 e^{-\kappa^2 \lambda} d\lambda \, \underline{f}^0(0) \underline{f}^0(0) \right\}$$

$$= \left\{ \int_0^{\tau_0} e^{-\kappa^2 \lambda} d\lambda \, I^2 \right\} \underline{f}^0(\tau_0) \underline{f}^0(\tau_0) \tag{15.16}$$

Finally, the synchronized function (15.12) is the asymptotic value in τ_0 of (15.16). Thus:

$$\underline{F}^{21}(\tau_0) \underset{\tau_0}{\sim} \left\{ \zeta^*_{12} I_{12} \right\} \underline{f}^0(\tau_0) f^0(\tau_0) \tag{15.17}$$

From (15.3) we can therefore select the solution

$$\frac{\partial \underline{f}^0}{\partial \tau_2} = L \left\{ \zeta^*_{12} I_{12} \right\} \underline{f}^0(\tau_0) \underline{f}^0(\tau_0) \tag{15.18}$$

We have thus proved that Bogolubov's kinetic equations are compatible with our perturbation equations.

B. Hydrodynamic Regime

In collaboration with McCune and Morse (18), a detailed calculation of the transients that precede the onset of the Navier-Stokes equations has been calculated with the local equilibrium model. Grad has independently considered the linearized Boltzmann equation.

We want to emphasize here that there are definite conditions, analogous to the principle of absence of parallel motions for the validity of these results. Thus, we must have

$$\frac{\partial \underline{u}'}{\partial \tau_0} \underset{\tau_0}{\sim} \frac{1}{\tau_0^{1+\eta}} \tag{15.19}$$

to insure the Eulerion behavior of $\frac{\partial \underline{u}^\circ}{\partial \tau_1}$ and

$$\underline{\mathbb{P}}_0 \longrightarrow \mathbb{I}\, k \underline{\rho}^\circ \underline{T}^\circ \tag{15.20}$$

fast enough, i.e. since

$$(\mathbb{P}_0)_{ij} = \int c_i c_j \underline{f}^\circ d\underline{c} = \int c_i c_j (\underline{f}^\circ - \underline{M}^\circ) d\underline{c} + \mathbb{I}_{ij} k \underline{\rho}^\circ \underline{T}^\circ \tag{15.21}$$

we must have

$$\int c_i c_j (\underline{f}^\circ - \underline{M}^\circ) d\underline{c} \sim \frac{1}{\tau_0^{1+\eta}} \tag{15.22}$$

We also note that the analogue of Bogolubov's problem of Section 9 is here the Enskog problem:

$$\underline{\rho}^k \longrightarrow 0, \quad \underline{u}^k \longrightarrow 0, \quad \underline{T}^k \longrightarrow 0, \quad \text{for } \tau_0 \longrightarrow \infty \tag{15.23}$$

for $K \neq 0$.

SECTION 16

TIME AVERAGE THEORY

The purpose of this section is to describe connection of our theory to Kirkwood's time averaging procedure and to show how Kirkwood's procedure can in fact be made to generate a complete hierarchy of time averages capable of giving the desired results to arbitrary accuracy.

We illustrate the scheme with the weak coupling expansion. We introduce

$$\langle \underline{\phi} \rangle_0 \equiv \lim_{T\to\infty} \frac{1}{T} \int_0^T \underline{\phi}(\tau_0, \tau_1, \ldots \tau_n, \ldots) d\tau_0 \tag{16.1}$$

Similarly,

$$\langle \underline{\phi} \rangle_1 \equiv \lim_{T\to\infty} \frac{1}{T} \int_0^T \underline{\phi}(\tau_0, \tau_1, \ldots \tau_n, \ldots) d\tau_1 \tag{16.2}$$

and so on.

From the weak coupling equation (7.32) we readily find:

$$\langle \underline{f}^0 \rangle_0 = \underline{f}^0 \tag{16.3}$$

For the two-body function, (7.37) gives

$$\langle \underline{F}^{20} \rangle_0 = \underline{f}^0 \underline{f}^0 \tag{16.4}$$

Therefore, in first order (7.33) reduces to

$$\frac{\partial \underline{f}^1}{\partial \tau_0} + \frac{\partial \underline{f}^0}{\partial \tau_1} = 0 \tag{16.5}$$

But, with

$$\left\langle \frac{\partial \underline{f}^1}{\partial \tau_0} \right\rangle_0 = 0 \tag{16.6}$$

we find

$$\frac{\partial \underline{f}^0}{\partial \tau_1} = 0 \tag{16.7}$$

whence

$$\langle \underline{f}^0 \rangle_1 = \underline{f}^0 \tag{16.8}$$

For the two-body function (7.38) gives

$$\begin{aligned} \langle \underline{F}^{21} \rangle_0 &= \lim_{T \to \infty} \frac{1}{T} \int_0^T \frac{1 - e^{-\kappa^2 \tau_0}}{\kappa^2} I^2 \underline{f}^0 \underline{f}^0 \, d\tau_0 \\ &= \frac{1}{\kappa^2} I^2 \underline{f}^0 \underline{f}^0 \end{aligned} \tag{16.9}$$

The singularity at $\kappa^2 = 0$ is determined without difficulty from the finite time behavior, thus

$$\langle \underline{F}^{21} \rangle_0 = \zeta^* I \underline{f}^0 \underline{f}^0 \tag{16.10}$$

Therefore, in second order, from (7.34)

$$\frac{\partial \underline{f}^2}{\partial \tau_0} + \frac{\partial \underline{f}'}{\partial \tau_1} + \frac{\partial \underline{f}^\circ}{\partial \tau_2} = L \underline{F}^{21} \qquad (16.11)$$

Averaging over the fast time scale

$$\frac{\partial \langle \underline{f}' \rangle_0}{\partial \tau_1} + \frac{\partial \underline{f}^\circ}{\partial \tau_2} = L \langle \underline{F}^{21} \rangle_0 = L \zeta^* I \underline{f}^\circ \underline{f}^\circ \qquad (16.12)$$

If we further average over τ_1 we have the Landau equation since

$$\left\langle \frac{\partial \langle \underline{f}' \rangle_0}{\partial \tau_1} \right\rangle_1 = 0 \qquad (16.13)$$

The method of extension thus provides the mathematical foundation of Kirkwood's procedure.

We have made of course many assumptions in this section about the behavior of the functions involved in regard to the properties of the time average. These assumptions in fact coincide with the previously derived conditions for the validity of the kinetic equations.

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(1) G. Sandri. ARAP Report No. 49, Part I, April 1963.

(2) A preliminary discussion of these results is incorporated in a paper written in collaboration with J. McCune and E. Frieman, Rarefied Gas Dynamics Symposium, Paris (1962).

(3) T. H. Dupree. Phys. Fluids 4, 696 (1961).

(4) I am especially indebted to Dr. T. Morse for discussions on the problem of this section.

(5) The same result is obtained by a completely different reasoning by Andrews. Andrews, Phys. Rev. 125, 1461 (1962).

(6) J. McCune and G. Sandri. ARAP Report No. 37, (1961). A more elaborate treatment is given by G. Sandri, ARAP Report No. 46, (1963).

(7) J. McCune, T. Morse, and G. Sandri. Rarefied Gas Dynamics Symposium, Paris, (1962). Also H. Grad's contribution will appear here.